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AUTOMORPHISMS OF THE LATTICE OF RECURSIVELY ENUMERABLE SETS. PART II: LOW SETS*

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1. Introduction

Informally, a subset A of N, the set of natural numbers, is *recursive* if there is an algorithm for computing its characteristic function, and *recursively enumerable* (r.e.) if there is an algorithm for enumerating its members. The existence of nonrecursive r.e. sets together with their frequent occurrence in many branches of mathematics has enabled them to play a crucial role in undecidability results beginning with Gödel's incompleteness theorem [5] and more recently in number theory and group theory. Matiyasevič [17, 18] answered Hilbert's 10th problem by showing unsolvability of Diophantine equations, specifically by proving that every r.e. set A is Diophantine, namely there is a polynomial $p(x, \bar{y})$ with integral coefficients such that $x \in A$ iff $(\exists \bar{y})[p(x, \bar{y}) = 0]$. Boone [3] and others proved that every r.e. degree is the degree of the word problem of a finitely presented group, thus generalizing the Boone–Novikov result [2, 20] that the word problem is unsolvable. Higman [6] showed a remarkable equivalence between r.e. sets and a purely algebraic property by proving that a finitely generated group G is embeddable in a finitely presented group iff for any finite set of generators $g_1, g_2, \ldots, g_k$ of G, the set of words on $g_1, g_2, \ldots, g_k$ equal to the identity in G is an r.e. set.

For sets $A, B \subseteq N$, A is *recursive in* (Turing reducible to) B, written $A \leq_T B$, if there is an algorithm for computing the characteristic function for A given that for B. Let $A \equiv_T B$ if $A \leq_T B$ and $B \leq_T A$. The *degree* of A, $\deg(A)$, is the equivalence class $\{B : B \equiv_T A\}$. A degree is r.e. if it contains an r.e. set. The study of the degrees in general was initiated by Kleene and Post [7], and Spector [38], although the study of the structure of the r.e. sets and their degrees was begun earlier by Post [21], who was the first to explain in an informal style the basic properties of r.e. sets and the role they played in Gödel's theorem.

The r.e. sets $\{W_n\}_{n \in N}$ under inclusion form a distributive lattice $\mathscr{E}$ whose complemented elements are precisely the recursive sets. Post's problem was to find an r.e. degree other than $\mathbf{0} = \deg(\emptyset)$ and $\mathbf{0}' = \deg(K)$, where K is the complete

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r.e. set $\{n: n \in W_n\}$. The classification of r.e. degrees amounts to the classification of unsolvable problems in mathematics, since for example the existence of infinitely many different r.e. degrees implies that there are infinitely many genuinely different unsolvable word problems for finitely presented groups, rather than just one. Post's program was to find a structural property on the complement $\bar{A}$ of an r.e. set A which guarantees $\emptyset <_T A <_T K$, and more generally to relate the $\mathscr{E}$-structure of A to its degree.

For $A \in \mathscr{E}$, define the principal filter $\mathscr{L}(A) = \{W: W \in \mathscr{E} \text{ and } A \subseteq W\}$. Much of the classification of r.e. sets has focused on these principal filters, since the principal ideals are all isomorphic because $\{W: W \in \mathscr{E} \text{ and } W \subseteq A\} \cong \mathscr{E}$ for every infinite $A \in \mathscr{E}$. Let $\mathscr{E}^*$ denote the quotient lattice of $\mathscr{E}$ modulo the ideal $\mathscr{F}$ of finite sets, A^* the equivalence class in $\mathscr{E}^*$ containing $A \in \mathscr{E}$, and $\mathscr{L}^* = \{A^*: A \in \mathscr{L}\}$ for any $\mathscr{L} \subseteq \mathscr{E}$. An r.e. set A is *maximal* of A^* is a coatom of $\mathscr{E}^*$, i.e., if $\mathscr{L}^*(A)$ is the 2 element Boolean algebra. The *jump* of $A \subseteq N$ is $A' = \{e: e \in W_e^A\}$, where $\{W_e^A\}_{e \in N}$ is an enumeration of the sets r.e. in A. The jump operator is well defined on degrees. An r.e. set is *high* if $A' \equiv_T \emptyset''$, the highest possible value for $\deg(A')$, and *low* if $A' \equiv_T \emptyset'$, the lowest possible value. An r.e. degree is high (low) if it contains a high (low) set.

A major advance in the program of relating the $\mathscr{E}$-structure of an r.e. set to its degree was Martin's result [16] that the degrees of maximal sets are precisely the high degrees. Later this was extended by replacing 'maximal' by various more general properties on $\bar{A}$ or $\mathscr{L}^*(A)$, such as '$\mathscr{L}^*(A)$ forms a Boolean algebra'.

In contrast to this variety of special structure for high r.e. sets, one might expect uniformity of structure for low r.e. sets because the recursive sets (which of course are all low) have very uniform structure. Namely, given any two infinite, coinfinite recursive sets A and B there is a recursive permutation of N which sends A to B, and hence

$$\text{there is an automorphism of } \mathscr{E} \text{ sending } A \text{ to } B, \tag{1.1}$$

and in particular,

$$\mathscr{L}(A) \cong \mathscr{E}. \tag{1.2}$$

Now suppose that A and B are any two infinite, coinfinite low r.e. sets. Of course, we cannot prove (1.1) because A and B may be recursive, simple, or neither. Some evidence for (1.2) was given by Robinson [22] in answer to a conjecture of Martin [16]. Robinson showed that such an A has a maximal superset. In [30] we invented a new method to prove that for any two maximal sets M_1, M_2, there is an automorphism of $\mathscr{E}$ carrying M_1 to M_2. By substantially extending this machinery we now prove (1.2) for any coinfinite low r.e. set. Surprisingly, this is the first example of any nonrecursive r.e. set satisfying (1.2). Furthermore, our proof does not require the full hypothesis that A is low, but only the weaker hypothesis that $\bar{A}$ is semi-low, namely $\{e: W_e \cap \bar{A} \neq \emptyset\} \leq_T \emptyset'$. Such r.e. sets A have arisen independently in other branches of recursion theory and computational complexity. In the latter area they have been classified [35] as exactly the nonspeedable r.e. sets, namely those with a 'fasest' algorithm modulo some recursive function. Since such sets A exist in every r.e. degree we see that (1.2) holds for some r.e. set in every r.e. degree.

Theorem 1.1. *If A is a coinfinite r.e. set such that $\{e: W_e \cap \bar{A} \neq \emptyset\} \leq_T \emptyset'$, then $\mathscr{L}(A) \cong \mathscr{E}$. Furthermore, the induced isomorphism $\Phi : \mathscr{L}^*(A) \cong \mathscr{E}^*$ is effective in the sense that there is a recursive permutation f of N such that for all e, $(\Phi(W_e \cup A))^* = W^*_{f(e)}$.*

This theorem and the maximal set automorphism theorem [33] demonstrate *uniformity* of structure of the r.e. sets just as the splitting theorem [26] and density theorem [27] of Sacks demonstrate unformity of structure of the r.e. degrees, whereas most recent results on r.e. sets and r.e. degrees emphasize their pathology. Theorem 1.1 applies to most r.e. sets commonly constructed in recursion theory (namely, those constructed by the finite injury priority method [28, Chapter 4]) because these sets are almost always low. For example, the original Friedberg [4]–Muchnik [19] solution to Post's problem automatically produces low sets, and the easiest known solution is to construct an r.e. set which is nonrecursive yet low. Indeed, producing a nonlow incomplete r.e. set necessitates infinitary positive requirements and usually involves an infinite injury priority construction.

Theorem 1.1 shows that for many diverse r.e. sets A, the principal filters $\mathscr{L}(A)$ are isomorphic to the filter $\mathscr{L}(\emptyset) = \mathscr{E}$ above the least element $\emptyset$ of $\mathscr{E}$. In contrast for the structure $(\boldsymbol{D}, \leq)$ of Turing degrees it is not known whether there is *any* degree $\boldsymbol{a} \neq \boldsymbol{0}$ such that $P_{\boldsymbol{a}}$ is isomorphic to $P_{\boldsymbol{0}}$ where $P_{\boldsymbol{a}}$ is the 'principal filter' $P_{\boldsymbol{a}} = \{\boldsymbol{b}: \boldsymbol{b} \in \boldsymbol{D} \text{ and } \boldsymbol{a} \leq \boldsymbol{b}\}$. Recently, Shore [32] has disproved the homogeneity conjecture by showing that there are many principal filters $P_{\boldsymbol{a}}$ not isomorphic (or even elementarily equivalent) to $P_{\boldsymbol{0}}$. Further classification of the principal filters of $\mathscr{E}$ should play an important role in resolving one of the major open questions, namely the decidability or undecidability of the elementary theory of $\mathscr{E}$ and finally its exact classification. For example, results such as [12, 11] on embedding certain lattices as principal idelas of $(\boldsymbol{D}, \leq)$ were used first to prove undecidability of the elementary theory of the degrees $(\boldsymbol{D}, \leq)$ and later by Simpson [32] to prove this theory recursively isomorphic to second order analysis.

This $\mathscr{E}$-structural resemblance between arbitrary low sets and recursive sets is analogous to the resemblance of a low r.e. degree $\boldsymbol{a}$ to $\boldsymbol{0} = \deg(\emptyset)$ in the structure of the r.e. degrees $(\boldsymbol{R}, \leq, \cup)$ with the order induced by $\leq_T$ and with $\deg(A) \cup \deg(B) = \deg(A \oplus B)$ where $A \oplus B = \{2x : x \in A\} \cup \{2x+1 : x \in B\}$. Let $\boldsymbol{a}|\boldsymbol{b}$ denote that $\boldsymbol{a}$ and $\boldsymbol{b}$ are incomparable under $\leq$. The splitting theorem of Sacks [26] for the r.e. degrees asserts

$$(\forall \boldsymbol{b} > 0)(\exists \boldsymbol{c})(\exists \boldsymbol{d})[\boldsymbol{b} = \boldsymbol{c} \cup \boldsymbol{d} \text{ and } \boldsymbol{c}|\boldsymbol{d}]. \tag{1.3}$$

R.W. Robinson proved [23, Corollary 9] that in (1.3), $\mathbf{0}$ can be replaced by any low r.e. degree $\boldsymbol{a}$ and the conclusion strengthened to include "$\boldsymbol{a} \leq \boldsymbol{c}$ and $\boldsymbol{a} \leq \boldsymbol{d}$". Lachlan later showed [10] that this cannot be done for every r.e. degree $\boldsymbol{a} < \mathbf{0}'$.

The plan of the paper is the following. In Section 2 we give a simple intuitive proof of Robinson's theorem that any coinfinite low set has a maximal superset. This proof has not yet appeared in the literature. Our purpose is to illustrate clearly in the easiest case how the hypothesis 'A is low' (or more generally '$\bar{A}$ is semi-low') is used, because this device appears later in a much more complicated setting in the proof of Theorem 1.1. In Section 3 we state the requirements necessary to establish Theorem 1.1, and we explain the strategy for meeting single requirements. Handling all the requirements simultaneously requires a much more complicated construction, which we present in Section 4. This construction depends upon certain rules, which are given in Section 5 together with the lemmas they are designed to achieve. In section 6 we verify that the construction succeeds.

This paper is intended to be self-contained so that familiarity with the automorphism paper [33] is not necessary. One can read Sections 1–3 with no reference to [33] at all. In Sections 4–6 it is necessary to give a much more complicated version of the Extension Theorem [33, Theorem 2.2]. Here we give complete definitions, notation, and the statements of all rules and lemmas. However, wherever a proof of a lemma is almost identical to one in [33] we simply refer to the proof there. Thus, to understand Sections 4–6, the reader should have a copy of [33] in hand for reference, but he need not have studied it in advance.

The relation of this proof with that in [33] is the following. If A is the given coinfinite low r.e. set, we shall construct by stages a 1–1 map ψ from $\bar{A}$ onto N which induces an isomorphism $\mathscr{L}(A) \cong \mathscr{E}$. In [33] to construct an automorphism Φ taking a maximal set M_1 to a maximal set M_2, the main tool was the Extension Theorem [33, Theorem 2.2] in which we constructed a 1–1 map ψ from M_1 to M_2 which helped induce Φ. In the latter case, since M_1 and M_2 were r.e. an element once enumerated in M_1 and put in dom(ψ) remained there forever. Now an element x apparently in $\bar{A} \cap \mathrm{dom}(\psi)$ at some stage may be enumerated in A at a later stage thereby leaving dom(ψ) forever and forcing ψ to be redefined. This forces the entire method to be substantially extended and made much more complicated than in [33]. The lowness of $\bar{A}$ is used to recursively 'guess' when certain elements of $\bar{A}$ will later enter A. The guessing must result in few enough mistakes so that ψ can be successfully defined. The main technical device not found in [33] for dealing with the case of dom ψ non-r.e. is discussed at the end of Section 5. The device was first developed to prove Theorem 1.1. It was then applied in a different way by M. Stob [39, 40] to generalize a result of D.A. Martin by proving that dense simplicity is not invariant under automorphisms of $\mathscr{E}$.

After these papers were written, W. Maass [14] defined the notion of an r.e. generic set, and proved that every such set is promptly simple and semi-low. An r.e. set is *promptly simple* if $\bar{A}$ is infinite and there exists a recursive function f and an index i, $W_i = A$, such that for all e

$$W_e \text{ infinite} \Rightarrow (\exists x)[x \in W_e \,\&\, x \in W_{i, f(\mu s[x \in W_{e,s}])}].$$

(Most constructions of simple sets automatically yield promptly simple sets.) Maass showed [14, Theorem 17] that the proof here could be combined with Extension Lemma [33, Theorem 2.2] to show that any two promptly simple semi-low sets are automorphic. Stob then showed that this could be applied to give an alternate proof of his theorem.

Unexplained notation and conventions can be found in Rogers [25]. Let $A =^* B$ denote that the symmetric difference of A and B is finite. Hence, $A =^* B$ just if $A^* = B^*$. Let $A \subseteq^* B$ denote that $A \cap \bar{B} =^* \emptyset$. A *recursive array* is a recursive sequence of r.e. sets. A *simultaneous enumeration* of a given recursive array $\{U_n\}_{n \in N}$ is a 1–1 recursive function g with range $\{\langle m, n\rangle : m \in U_n\}$. Thus, at each stage s, $g(s) = \langle m, n\rangle$ causes *one* element m to be enumerated in *one* r.e. set U_n. Fixing g, let $U_{n,s}$ denote those elements enumerated in U_n by stage s, and

$$U_n \setminus U_m = \{x : (\exists s)[x \in U_{n,s} - U_{m,s}]\},$$

those elements appearing in U_n before U_m. (The notation $X \setminus Y$ should not be confused with $X - Y$ which denotes $X \cap \bar{Y}$.) We let $X \searrow Y$ denote $(X \setminus Y) \cap Y$. We use these notations only when we have in mind a *particular* simultaneous enumeration.

A *standard enumeration* (of the r.e. sets) is a simultaneous enumeration of $\{U_n\}_{n \in N}$, where the latter is some acceptable numbering of the r.e. sets. From now on we fix a standard enumeration of our acceptable numbering $\{W_n\}_{n \in N}$, thus yielding the double array $\{W_{n,s} : n, s \in N\}$ of finite approximations. It will be convenient later to introduce other recursive arrays $\{U_n\}_{n \in N}$ with simultaneous enumerations. With respect to such $\{U_{n,s} : n, s \in N\}$, we define the *e-state of an element x at stage s*,

$$\sigma(s, e, x) = \{i : i \leqslant e \text{ and } x \in U_{i,s}\}.$$

The e-states, being finite sets, are identified with their characteristic functions which are ordered lexicographically as usual (see Rogers [25, p. 235]). We say e-state σ is *higher* than e-state τ if its characteristic function precedes that of τ.

2. Low sets and maximal supersets

In this section we give a simple intuitive proof of Robinson's theorem that any coinfinite low r.e. set A has a maximal superset M. We explain carefully the device upon which the proof turns since we need a more complicated version of it later for our main theorem.

2.1. Motivation

We assume familiarity with the usual Yates construction [25, p. 235] of a maximal set M. Let M_s denote those elements enumerated in M by the end of stage s of the construction. Briefly, M is defined using movable markers $\{\Lambda_e\}_{e\in N}$ such that the eth member of $\bar{M}_s$ is defined to be Λ_e^s, the position of marker Λ_e at the end of stage s, and such that Λ_e moves to maximize the e-state of Λ_e^s with respect to $\{W_e\}_{e\in N}$.

Now assume that we are given an r.e. set A and a recursive enumeration $\{A_s\}_{s\in N}$ of A. To insure that $\bar{M}\subseteq\bar{A}$ we must arrange that each marker Λ_e comes to rest on some $x\in\bar{A}$.

2.2 The strategy for a single marker

For simplicity we consider only marker Λ_0. Now if $\Lambda_0^s=x\in W_{0,s}-A_s$, then perhaps at some later stage $t>s$, $x\in A_t$ so that Λ_0 is forced to move. This may happen infinitely often causing Λ_0 not to settle and thus $\bar{M}=\emptyset$. For Λ_0 to select from those *apparently* desirable elements $x\in W_{0,s}-A_s$ those which are *truly* desirable ($x\in\bar{A}$) we must use the lowness of A, or more precisely, the semi-lowness of $\bar{A}$, namely, $H_{\bar{A}}=\{e\colon W_e\cap\bar{A}\neq\emptyset\}\leq_T\emptyset'$. For such an $\bar{A}$ we have a 0–1-valued function $\hat{h}\equiv_T\emptyset'$ such that for all e

$$W_e\cap\bar{A}\neq\emptyset\Leftrightarrow\hat{h}(e)=1. \tag{2.1}$$

By the Limit Lemma [29, Theorem 2] there is a recursive function $h(e,s)$ such that for all e

$$\lim_s h(e,s)\text{ exists}\quad\text{and}\quad=\hat{h}(e). \tag{2.2}$$

We use h as a kind of 'oracle' for $\bar{A}$ to determine whether a given element x which we want for Λ_0 is really in $\bar{A}$. At stage s we move Λ_0 to x only if the oracle 'says' $x\in\bar{A}$ (i.e., $h(\theta(0),s)=1$ for a certain r.e. set $W_{\theta(0)}$ containing x). Now if $x\in A_t$ for $t>s$, then the oracle 'lied' to us, but (2.2) and very careful definition of $W_{\theta(0)}$ insure that the oracle lies at most finitely often.

Now define the r.e. set $Y_0\subseteq W_0$ by

$$Y_{0,s+1}=\begin{cases}Y_{0,s} & \text{unless } Y_{0,s}\subseteq A_s \quad\text{and}\quad (\exists x)[x\in W_{0,s+1}-M_s],\\ Y_{0,s}\cup\{x'\} & \text{where } x'=\mu x[x\in W_{0,s+1}-M_s],\\ & \text{otherwise.}\end{cases}$$

By the recursion theorem we may assume that we have in advance an index $\theta(0)$ such that $W_{\theta(0)}=Y_0$. At stage s, given a candidate $x\in Y_{0,s}-A_s$, define

$$t'=(\mu t\geq s)[x\in A_t\text{ or }h(\theta(0),t)=1]. \tag{2.3}$$

(Note that t' exists because if $x\in\bar{A}$, then $x\in W_{\theta(0)}\cap\bar{A}$ so $\hat{h}(\theta(0))=\lim_s h(\theta(0),s)=1$.) Now move Λ_0 to x just if $x\notin A_{t'}$ and $h(\theta(0),t')=1$, in which case we say that the oracle function h permits Λ_0 to move at stage s.

The crucial point is that

$$|Y_{0,s}-A_s|\leq 1 \tag{2.4}$$

because each candidate $x\in Y_{0,s}-A_s$ must enter A before we unumerate a new candidate in Y_0. Now suppose that Λ_0 moves infinitely often. Then

$$W_{\theta(0)}\cap\bar{A}=\emptyset \tag{2.5}$$

because by (2.4), Λ_0 moves to every element of $Y_0=W_{\theta(0)}$, and each of these elements is later enumerated in A. But Λ_0 moves only when permitted, so $h(\theta(0),t)=1$ for infinitely many t. Thus $\hat{h}(\theta(0))=1$ by (2.2), but (2.5) contradicts (2.1).

2.3. *Test markers*

Intuitively, we should picture the unique $x\in Y_{0,s}-A_s$ (if x exists) as the position of a 'test' marker Γ_0 to which Λ will move if permitted. Thus if we specify how to move Γ_0 we can define Y_0 to be the set of its positions. (Note that even though Λ_0 moves finitely often, the test marker Γ_0 may move infinitely often, and indeed does so if $|W_0\setminus A|=\infty$ but $|W_0-A|=0$.) The method of Section 2.2 gives us a recursive oracle which at stage s answers the question "Is the current position of Γ_0 in A?", and which lies at most finitely often.

The above method required (specifically for (2.5)) that Γ_0, once associated with an element x, remained associated with x and not with any new element unless x was first enumerated in A. Namely, we required that

$$Z_{0,s}\subseteq Z_{0,s+1}\cup A_{s+1}, \tag{2.6}$$

where $Z_{0,s}$ denotes the positions occupied by Γ_0 *during* stage s. With several test markers, say Γ_0 for Λ_0 and Γ_1 for Λ_1, we can no longer guarantee (2.6). For example, suppose that Γ_0 rests on x_0 and Γ_1 on x_1 at stage s and that x_0 is enumerated in A at stage $s+1$. Then Γ_0 may move to x_1, forcing Γ_1 to move without its former position x_1 being enumerated in A. Thus we say that Γ_1 is *injured at stage* $s+1$ if $Z_{1,s}\not\subseteq Z_{1,s+1}\cup A_{s+1}$, where $Z_{e,s}$ represents the position (or positions) of marker Γ_e during stage s. For marker Γ_1 at stage s we let $W_{\theta(1,s)}$ consist only of those positions of Γ_1 since the last stage $v\leq s$ at which Γ_1 was injured, namely

$$W_{\theta(1,s)}=\bigcup\{Z_{1,t}: v\leq t\}.$$

Now if Γ_1 is injured only finitely often, then $\lim_s\theta(1,s)=\theta(1)$ exists and $W_{\theta(1)}$ suffices for Γ_1 just as $W_{\theta(0)}$ did earlier for Γ_0.

2.4. *The role of the recursion theorem*

Applying the recursion theorem poses no problem even though we define during the construction infinitely many r.e. sets $\{W_{\theta(e,s)}\}_{s\in N}$ whose indices $\theta(e,s)$ we must know in advance.

Given W_j let $W_j^{\langle y,z\rangle}=\{x: \langle x, y, z\rangle \in W_j\}$, and define a recursive function g such that $g(j, y, z)=W_j^{\langle y,z\rangle}$. During the construction we define a single r.e. set $W_{f(j)}$ by setting $W_{f(j)}^{\langle e,s\rangle}=\bigcup\{Z_{e,t}: v\leq t\}$, the positions of marker Γ_e since the last stage $v\leq s$ at which Γ_e was injured. At stage $s+1$, the oracle function h 'permits' Λ_e to move to Γ_e if $h(g(j, e, s), s)=1$. At the end of the construction we have a total recursive function f. By the recursion theorem, f has a fixed point j^* such that $W_{f(j^*)}=W_{j^*}$. Now set $W_{\theta(e,s)}=W_{f(j^*)}^{\langle e,s\rangle}=W_{j^*}^{\langle e,s\rangle}=W_{g(j^*,e,s)}$.

2.5. Low sets have maximal supersets

Theorem 2.1 (R.W. Robinson [22]). *If A is a coinfinite r.e. set such that $\bar{A}$ is semi-low (i.e., $H_{\bar{A}}=\{e: W_e\cap\bar{A}\neq\emptyset'\}\leq_T\emptyset'$), then A has a maximal superset M.*

Proof. Since $\bar{A}$ is semi-low, fix a recursive function h satisfying (2.1) and (2.2) above. For each $e\in N$, we have a marker Λ_e which will come to rest on the eth member of $\bar{M}$, and also 2^{e+1} 'test' markers Γ_e^σ, one for each e-state σ. Assume the r.e. sets have been indexed so that $W_1=A$. Let $A_s(M_s)$ consist of those elements enumerated in A (respectively, M) by the end of stage s in the following construction. (In general, A_s will contain more elements than $W_{1,s}$ since the oracle produces 'speed-up' of the enumeration $\{W_{1,s}\}_{s\in N}$ of A.) Let $Z_{e,s}^\sigma$ denote the set of all positions of marker Γ_e^σ during stage s.

Stage 0. For each e and σ, appoint Λ_e to e.

Stage $s+1$. This will consist of $s+2$ substages, one for each $e, 0\leq e\leq s$, and a final substage $e=s+1$. At each substage $e, 0\leq e\leq s$, marker Λ_e will be considered, and M_s will be defined at substage $s+1$. Enumerate in A all $x\in W_{1,s+1}$.

Substage e (for $0\leq e\leq s$). For every i, let m_i be the current position of marker Λ_i, and let $m_{-1}=-1$.

Step 1. Let B_e consist of those elements $y>m_{e-1}$ not yet enumerated in either A or M.

Step 2. If $m_e\notin B_e$, let $x=\mu y[y\in B_e]$, and let $\sigma=\sigma_0$, the empty e-state. Go to step 4.

Step 3. If $m_e\in B_e$, choose the highest e-state σ such that for some $x\in B_e$

$$m_e<x \;\&\; \sigma(e, m_e, s)<\sigma(e, x, s)=\sigma. \tag{2.7}$$

If no such σ and x exist, appoint to m_e any Γ_e^σ appointed to it at the end of stage s, and go to substage $e+1$. Otherwise choose x minimal for σ, and go to step 4.

Step 4. Appoint Γ_e^σ to x (i.e., enumerate x in $Z_{e,s+1}^\sigma$). Let v be the last stage $t+1\leq s+1$ at which Γ_e^σ was injured, namely,

$$Z_{e,t}^\sigma\nsubseteq Z_{e,t+1}^\sigma\cup A_{t+1}. \tag{2.8}$$

We can tell immediately whether Γ_e^σ is injured at $s+1$ because any $y\in Z_{e,s}^\sigma$ which will enter $Z_{e,s+1}^\sigma$ has already done so. If no such t exists, set $v=0$. Using the recursion theorem as in Section 2.4 define the recursive function θ by

$$W_{\theta(e,\sigma,s)} = \bigcup\{Z^{\sigma}_{e,t}: t \geq v\},$$

and choose

$$t' = (\mu t \geq s)[x \in W_{1,t} \text{ or } h(\theta(e,\sigma,s),\ ) = 1].$$

Step 5. If $x \in W_{1,t'}$ enumerate x in A and return to step 1.

Step 6. If $x \notin W_{1,t'}$ and $h(\theta(e,\sigma,s),t') = 1$, choose the unique n such that $x = m_{e+n}$. For each $i \geq e$ move marker Λ_i to m_{i+n}.

Substage $s+1$. Move the markers Λ_e, $e > s$, so that they are assigned, in order, to the numbers greater than the position of Λ_s which are not yet in A or M. Enumerate in M all elements not currently associated with some marker Λ_e, $e \in N$.

This ends the construction.

Notice that substage e above must terminate. If not, then steps 2 and 5 are repeated infinitely often, because step 3 can apply only finitely often at any stage s since $\bigcup\{W_{e,s}: e \in N\}$ is finite. But then $\Gamma^{\sigma_0}_e$ is eventually appointed to some $x \in \bar{A}$ because $\bar{A}$ is infinite while A_s and M_s are finite. However, step 5 cannot apply to any $x \in \bar{A}$.

Let $m_{e,s}$ denote the position of marker Λ_e at the end of stage s.

Lemma 1. *For all* $e, \lim_s m_{e,s} = m_e$ *exists and* $m_e \in \bar{A}$. (*Thus* $\bar{M}$ *is infinite and* $A \subseteq M$.)

Proof. The construction guaranteed at the end of each stage s that $m_{e,s} \in \bar{A}_s$. Hence, it suffices to prove that $\lim_s m_{e,s}$ exists. By induction on e, fix e and choose s_0 such that $m_{i,s} = m_i$ for all $i \leq e$ and $s \geq s_0$. Therefore, after stage s_0 marker Λ_i is never moved at any substage $i < e$.

Now assume for a contradiction that marker Λ_e moves infinitely often. But if Λ_e moves at stage $s > s_0$, then step 6 of substage e applies via some marker Γ^{σ}_e. Let σ be the highest σ' such that step 6 applies infinitely often to $\Gamma^{\sigma'}_e$. Choose $s_1 > s_0$ such that for every e-state σ' higher than σ, step 6 of substage e does not apply to $\Gamma^{\sigma'}_e$ at any stage $s \geq s_1$. Suppose $s+1 \geq s_1$ and $x \in Z^{\sigma}_{e,s}$. By the construction, either $x \in A_s$ or else Γ^{σ}_e was appointed to $m_{e,s}$ at the end of stage s and so $x = m_{e,s}$. In the latter case, step 6 cannot apply at $s+1$, since $s+1 \geq s_1$. Thus either Γ^{σ}_e is appointed to $m_{e,s+1} = m_{e,s}$ via step 3 or else step 2 applies and $m_{e,s} \in A_{s+1}$. We have thus shown that Γ^{σ}_e cannot be injured at $s+1 \geq s_1$. Thus $\theta(e,\sigma,s) = \theta(e,\sigma,s_1)$ for all $s \geq s_1$. Let $\theta(e,\sigma) = \lim_s \theta(e,\sigma,s)$. If Γ^{σ}_e were ever assigned to a member of $\bar{A}$ after s_1, then Γ^{σ}_e would be assigned to this same number necessarily $m_{s,e}$, at every subsequent stage s. Thus

$$W_{\theta(e,\sigma)} \cap \bar{A} = \emptyset \tag{2.9}$$

because marker Λ_e moves infinitely often.

On the other hand, A_e moves at step 6 only if h permits. Hence,

$$(\exists^\infty s)(\exists\, t \geq s)[h(\theta(e,\sigma,s),t)=1];$$

this implies

$$(\exists^\infty t)[h(\theta(e,\sigma),t)=1].$$

Hence, $\hat{\imath}(\theta(e,\sigma))=1$ by (2.2) but then (2.9) contradicts (2.1).

Lemma 2. *M is maximal.*

Proof. Assume not. For each i let m_i be the final position of Λ_i. Let e be minimal such that W_e splits $\bar{M}$. Choose $i>e$ such that all m_k, $k\geq i$, are in the same final $(e-1)$-state, and $m_i \notin W_e$. Choose the least $j>i$ such that $m_j \in W_e$. Choose s such that the e-state of m_k, for all $k\leq j$, has settled by stage s. Now at step 3 substage i of stage $s+1$, Γ_i^σ is appointed to m_j, where $\sigma=\sigma(i,m,s)$. But step 5 cannot apply since $m_j \in \bar{A}$. Thus step 6 must apply so Λ_i moves to m_j, a contradiction.

Bennison and Soare [1, Theorem 4.7] have extended this theorem by obtaining the same conclusion under the weaker hypothesis '$\bar{A}$ is semi-low$_{1.5}$', namely

$$\{e: W_e \cap \bar{A} \text{ is infinite}\} \leq_m \{e: W_e \text{ is infinite}\}.$$

3. The requirements

3.1. *Specifying the isomorphism*

For any set $S\subseteq N$ let $\mathscr{E}_S$ denote the lattice with members $\{W_n \cap S : n \in N\}$, and the usual operations of $\cup$ and $\cap$. Let $\mathscr{E}_S^*$ denote $\mathscr{E}_S$ modulo $\mathscr{F}$, the ideal of finite sets. Now if A is r.e. it is easy to see that $\mathscr{L}(A)\cong\mathscr{E}_{\bar{A}}$ via the correspondence $W_e \cup A \leftrightarrow W_e \cap \bar{A}$.

From now on fix a coinfinite r.e. set A such that $\bar{A}$ is semi-low, and assume that the r.e. sets have been renumbered so that $W_1 = A$. To prove Theorem 1.1 it suffices to construct an isomorphism Φ from $\mathscr{E}_{\bar{A}}^*$ to $\mathscr{E}^*$. We do this by defining a 1–1 map ψ from $\bar{A}$ onto N which induces Φ, i.e., we define $\Phi(W_n \cap \bar{A}) =^* \psi(W_n \cap \bar{A})$. We shall also specify recursive functions F and G such that

$$\psi(W_n \cap \bar{A}) =^* W_{F(n)} \quad \text{and} \quad \psi^{-1}(W_n) \cup A =^* W_{G(n)} \cup A. \tag{3.1}$$

This guarantees that the corresponding isomorphism $\Phi : \mathscr{E}_{\bar{A}}^* \cong \mathscr{E}^*$ will be effective in the sense of Theorem 1.1; namely, there is a permutation f of N such that $\psi(W_e \cap \bar{A}) =^* W_{f(e)}$. The purpose of ψ is to guarantee that Φ preserves inclusion, while F and G guarantee that (modulo A) ψ and ψ^{-1} take r.e. sets to r.e. sets.

To meet (3.1), we shall enumerate two recursive arrays $\{U_n\}_{n\in N}$ and

$\{V_n\}_{n\in N}$ such that for all n

$$U_n =^* W_n =^* V_n, \tag{3.2}$$

and we shall simultaneously enumerate recursive arrays $\{\hat{U}_n\}_{n\in N}$ and $\{\hat{V}_n\}_{n\in N}$ such that

$$\psi(U_n \cap \bar{A}) =^* \hat{U}_n \tag{3.3}$$

and

$$\psi^{-1}(V_n) \cup A =^* (\hat{V}_n \cup A). \tag{3.4}$$

Thus we can meet (3.1) by setting $W_{F(n)} = \hat{U}_n$, $W_{G(n)} = \hat{V}_n$. The r.e. sets U_n, V_n, $\hat{U}_n$, $\hat{V}_n$ will be defined during the construction in Sections 4 and 5, but the map ψ will not be specified explicitly until Section 6.4. For any set X we let X_s denote the elements enumerated in X by the end of stage s in our construction.

In general, ψ will not be recursive or even recursive in ϕ', but to grasp the difficulties in meeting the requirements (3.3) and (3.4), let us first consider a naive picture where ψ is specified as the limit of maps ψ_s such that ψ_s is a 1–1 recursive map from $\bar{A}_s$ onto N, and ψ_s attempts to map $U_{n,s}$ to $\hat{U}_{n,s}$ and $\hat{V}_{n,s}$ to $V_{n,s}$ (modulo A). The major difficulty in attempting to construct ψ is that an element $x \in \operatorname{dom} \psi_s$ may be enumerated in A_{s+1} thereby leaving $y = \psi_s(x)$ with no appropriate preimage. In the next two subsections we show how the semi-lowness of $\bar{A}$ can be used to resolve the difficulty associated with meeting a single requirement of the form (3.3) or (3.4). These examples give only a small glimpse into the whole construction and are somewhat misleading because the major difficulties in the proof arise from attempting to simultaneously meet *all* the requirements, just as in the usual infinite injury constructions such as the Sacks Density Theorem.

3.2. *A single requirement of the form* (3.3)

To aid in understanding the later machinery, imagine two copies of N, the 'left-hand side' $\{n : n \in N\}$, where we are building $\operatorname{dom} \psi$, U_n, $\hat{V}_n$, and the 'right-hand side', $\{\hat{n} : n \in N\}$ where we are building $\operatorname{rng} \psi$, $\hat{U}_n$, and V_n. Suppose that $x \in \bar{A}_s = (\bar{W}_{1,s})$, $\psi_s(x) = \hat{y}$, and $x \in U_{n,s+1} - U_{1,s}$. To achieve (3.3) we might then enumerate $\hat{y}$ in $\hat{U}_n$. However, x may be enumerated in A at some later stage $t > s$, so we are forced to redefine $\psi_t^{-1}(\hat{y})$. Since $\hat{y} \in \hat{U}_{n,t}$ we must choose for $\psi^{-1}(\hat{y})$ some $x' \in U_{n,t}$ but no such x' may exist. If this is repeated over infinitely many x we may have $\hat{U}_n$ infinite (because $\limsup_s |U_{n,s} - A_s| = \infty$) but $U_n - A =^* \emptyset$ so (3.3) fails.

Since $\bar{A}$ is semi-low, we can overcome this difficulty by using the test markers as in Section 2 to give a different enumeration $\{U_{1,s}\}_{s\in\omega}$ of $A = W_1 = U_1$ such that

$$U_n - U_1 =^* \phi \Rightarrow U_n \setminus U_1 =^* \emptyset \tag{3.5}$$

and thus

$$U_n - U_1 =^* \phi \Rightarrow \hat{U}_n =^* \emptyset.$$

Roughly, we have test markers $\{\Gamma_{n,i}\}_{i\in N}$, and corresponding r.e. sets $W_{\theta(n,i,s)}$ as in Section 2. Let h be a recursive function satisfying (2.1) and (2.2). When $x \in W_{n,s+1} - W_{n,s}$, and $x \in \bar{U}_{1,s}$ choose the first i such that $\Gamma_{n,i}$ is unassigned, assign $\Gamma_{n,i}$ to x and let

$$t' = (\mu t > s)[x \in W_{1,t} \text{ or } h(\theta(n, i, s), t) = 1].$$

If the first clause holds, enumerate x in U_1 and in either case enumerate $x \in U_n$. The new simultaneous enumerations $\{U_{1,s}\}_{s\in N}$, $\{U_{n,s}\}_{s\in N}$ satisfy (3.5).

Notice that since the oracle may lie to us finitely often, it is possible that $U_n - A = \emptyset$ while $\hat{U}_n \neq \emptyset$ but is still finite. This suggests that we should think of ψ not as a 1–1 map but rather a finite-to-one (or one-to-finite) correspondence. This still suffices to meet (3.3) and (3.4) as we shall see in Section 6.4.

3.3. *A single requirement of the form* (3.4)

Suppose that $x \in \bar{A}_s$ $(= \bar{W}_{1,s})$, $\psi_s(x) = \hat{y}$, and $\hat{y} \in V_{n+1,s} - V_{n,s}$. Then to achieve (3.4) we might want to enumerate $x \in \hat{V}_n$ at stage s. However x may be enumerated in A at some later stage $t > s$. This causes no immediate problem for $\hat{y}$ since we can simply choose a fresh $x' \notin \operatorname{dom} \psi_t$, redefine $\psi_{t+1}^{-1}(\hat{y}) = x'$, and enumerate x' in $\hat{V}_n$.

However, if this occurs for infinitely many $\hat{y}$, we may eventually enumerate *every* element x of $\bar{A}$ into $\hat{V}_n$. This poses a problem if there is any $\hat{z} \in \bar{V}_n$, since there is no $x \in \bar{A} - \hat{V}_n$ which can be assigned to $\psi^{-1}(\hat{z})$. Suppose $\psi_{s_1}(w_1) = \hat{z}$, and meanwhile several elements $x \in \operatorname{dom} \psi_{s_1}$ are enumerated in $\hat{V}_n$. Then w_1 is enumerated in A_{s_2} at stage $s_2 > s_1$, so we must choose $w_2 \notin A_{s_1} \cup \hat{V}_{n,s_2}$ and set $\psi_{s_2}(w_2) = \hat{z}$. Meanwhile, more elements x of $\bar{A}$ are enumerated in $\hat{V}_n$ before w_2 is enumerated in A_{s_3}, at some stage $s_3 > s_2$, and so on. If each true element $x \in \bar{A}$ is enumerated in $\hat{V}_n$ before it becomes $\psi_s^{-1}(\hat{z})$, then $\lim_s \psi_s^{-1}(\hat{z})$ fails to exist. If this happens for infinitely many $\hat{z} \in \bar{V}_n$, then (3.3) may fail because $\bar{V}_n$ is infinite while $\hat{V}_n \supseteq \bar{A}$.

We can overcome this problem with a test marker $\Gamma_{\hat{z}}$ associated with $\hat{z} \in \bar{V}_n$. When $\psi_s(w) = \hat{z}$ and $w \in A_t - A_s$ for some $t > s$, we temporarily halt enumeration of $\hat{V}_n$ and repeatedly associate $\Gamma_{\hat{z}}$ with new elements $w \notin A_t \cup \hat{V}_{n,t}$ until the oracle asserts that one of these is indeed in $\bar{A}$. This search must terminate because $|\bar{A}| = \infty$ and $\hat{V}_{n,s}$ is finite, so $\Gamma_{\hat{z}}$ must eventually find a true element of $\bar{A}$.

These two examples are somewhat misleading in their over simplification, but they accurately reflect how the author first approached the problem. In the following full proof there will be not two kinds of markers, but only one kind which takes into account the 'full e-state' including both the U_n and $\hat{V}_n$ sets for $n \leq e$. Case 2(a) of the constructions will be the formal analogue of this informal description.

4. The construction

4.1. Preliminaries

Wherever possible we follow the framework of [33] although we have made some further simplifications and organizational changes, and have also incorporated some due to Stob [40]. We assume that we are given a recursive function g which simultaneously enumerates $\{W_n\}_{n\in N}$ and $\{\hat{W}_n\}_{n\in N}$ where $\{W_n\}_{n\in N}$ and $\{\hat{W}_n\}_{n\in N}$ are standard indexings of the r.e. sets. We shall also assume for convenience that $W_0=\hat{W}_0=N$, $W_1=A$, and g enumerates $x\in W_0(\hat{W}_0)$ before enumerating $x\in W_e(\hat{W}_e)$ for any $e>0$. Furthermore, we assume that if $x\in W_e$, $e>1$, then g enumerates x in $W_e(\hat{W}_e)$ at infinitely many different stages, but g enumerates W_0 and W_1 ($\hat{W}_0$ and $\hat{W}_1$) without repetitions. We shall enumerate in stages recursive arrays of r.e. sets $\{U_n\}_{n\in N}, \{V_n\}_{n\in N}, \{\hat{U}_n\}_{n\in N}, \{\hat{V}_n\}_{n\in N}$ and construct a 1–1 map ψ from $\bar{A}$ to N satisfying the requirements (3.2), (3.3) and (3.4).

4.2. The machines and notation

We shall, as in [33], present the proof using pinball machines M and $\hat{M}$. The *pinball machine* M is shown in Fig. 1 and $\hat{M}$ is identical except that each symbol X is replaced by $\hat{X}$. Two copies of N, $\{n\}_{n\in N}$ and $\{\hat{n}\}_{n\in N}$ act as *balls* of M and $\hat{M}$. A ball x ($\hat{x}$) initially *enters* M ($\hat{M}$) from hole H_1 ($\hat{H}_1$). It then proceeds along the

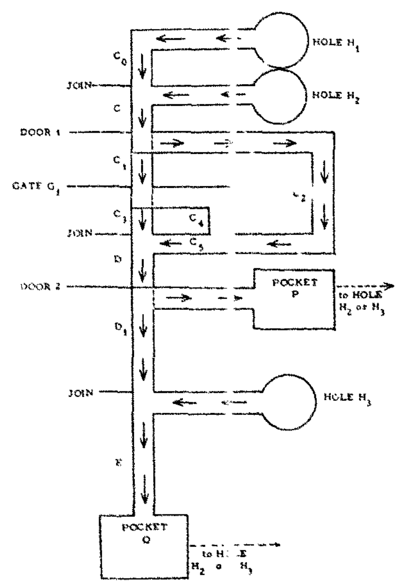

Fig. 1. Diagram of machine M.

surface of the machine, that portion of the machine covered by arrows, until it reaches a pocket. From a pocket, x ($\hat{x}$) may be placed above hole H_2 or H_3 from which it can *re-enter* the surface of the machine. The sequence of consecutive stages beginning when x ($\hat{x}$) enters (or re-enters) the surface of M ($\hat{M}$) and ending when x ($\hat{x}$) enters a pocket is called a *play* for x ($\hat{x}$). The motion of x ($\hat{x}$) at each stage of the play is specified by the *rules* of Section 5.

As soon as an element x of M is enumerated in U_1 ($= A$) it is removed from M forever. Let M_s (M_ω) denote those elements in M at the end of stage s (at the end of the construction) including those elements above holes, and likewise $\hat{M}_s$, $\hat{M}_\omega$. We shall arrange that $M_\omega = \bar{A}$ and $\hat{M}_\omega = N$.

If $x \in M_s$, the *full e-state of x at stage s*, denoted $\nu(s, e, x)$, is the triple $\langle e, \sigma(s, e, x), \tau(s, e, x)\rangle$ where $\sigma(s, e, x)$ and $\tau(s, e, x)$ denote the e-states of x at s measured with respect to $\{U_n\}_{n\in N}$ and $\{\hat{V}_n\}_{n\in N}$, respectively. We shall use e-states as finite sets of integers $\leq e$. If $\hat{x} \in \hat{M}_s$, $\nu(s, e, \hat{x})$ is $\langle e, \sigma(s, e, \hat{x}), \tau(s, e, \hat{x})\rangle$ where $\sigma(s, e, \hat{x})$ and $\tau(s, e, \hat{x})$ are e-states measured with respect to $\{\hat{U}_n\}_{n\in N}$ and $\{V_n\}_{n\in N}$. If σ, τ are e-states and $\nu = \langle e, \sigma, \tau\rangle$ a full e-state, $[\sigma]_i = \sigma \cap \{0, 1, \ldots, i\}$ and $[\nu]_i = \langle i, [\sigma]_i, [\tau]_i\rangle$. Let $\nu_{-1} = \langle -1, \emptyset, \emptyset\rangle$. Let $\nu = \langle e, \sigma, \tau\rangle$ and $\nu' = \langle e', \sigma', \tau'\rangle$. We say ν' *extends* ν, written $\nu \lessdot \nu'$, if $e \leq e', [\sigma']_e = \sigma$, and $[\tau']_e = \tau$. We say $\nu \leq \nu'$ if $e = e', \sigma \subseteq \sigma'$, and $\tau \supseteq \tau'$. Let $\nu \leq^\tau \nu'$ ($\nu \leq^\sigma \nu'$) denote $\nu \leq \nu'$ and $\tau = \tau'$ ($\sigma = \sigma'$). If $\mathcal{S}$ is a set of full e-states, let $\mathcal{S}[\nu_0] = \{\nu: \nu \in \mathcal{S}$ and $\nu_0 \lessdot \nu\}$. If $\nu = \langle e, \sigma, \tau\rangle$, the *length of* ν, $|\nu|$, is e. A *track* is a section of the surface of the machine between any two of the following: door, gate, pocket or join.

If X is a track of M, let $\mathcal{S}_s(X)$ be the sequence $\{\nu(s, e, x): e \leq x$ and x enters track X at stage $s\}$ arranged in order of e. Let $\mathcal{S}(X)$ be the concatenation of the $\mathcal{S}_s(X)$, $s \in N$. Such a sequence of full e-states is called *stream* X. If $\nu \in \mathcal{S}(X)$ infinitely often, we write $\nu \in \mathcal{S}(X)$ *i.o.* $\mathcal{S}(\hat{X})$ for $\hat{X}$ a track of $\hat{M}$ is defined similarly. We say a stream X *covers* (*dual covers*, *τ-exactly covers*, *exactly covers*) ν if some $\nu' \in \mathcal{S}(X)$ i.o., where $\nu \leq \nu'$ ($\nu' \leq \nu$, $\nu \leq^\tau \nu'$, $\nu = \nu'$). Stream X *covers* stream Y if X covers every ν such that $\nu \in \mathcal{S}(Y)$ i.o.

We now describe the role of pockets P and $\hat{Q}$; $\hat{P}$ and Q are similar. To construct ψ satisfying (3.3) and (3.4) we wish to match elements $x \in M$ and $\hat{y} \in \hat{M}$ which are in the same e-state. Pocket $\hat{Q}$ will consist of elements $\hat{y}$ needing mates. We shall attempt to choose mates for the elements of $\hat{Q}$ from those elements x entering track D. We can choose such an x as a mate for $\hat{y}$ (and put x in pocket P) at stage $s+1$ if $\nu(s, d, x) = \nu(s, d, \hat{y})$ where d is a certain function of s and $\hat{y}$. We allow $d(s, y)$ to be small when few x appear in states $\nu \lessdot \nu(s, d, \hat{y})$, thereby making it easier to choose a mate for $\hat{y}$. However, to insure that each requirement (3.3), (3.4) is respected by almost every $\hat{y}$ we must arrange that for each n there are at most finitely many $\hat{y}$ such that $d(s, \hat{y}) < n$ for some s. The function d is defined using a crucial sequence $\mathcal{M}$ of full e-states defined in Section 5.5. As more elements x appear on track D, more states ν are added to $\mathcal{M}$, allowing $d(s, \hat{y})$ to be large for most $\hat{y}$. But when a temporary mate x for $\hat{y}$ *leaves* pocket P, say because x is enumerated in U_n for some $n < d(s, \hat{y})$, then various ν are later *excluded* from $\mathcal{M}$, thereby tending to decrease $d(s, \hat{y})$ and making it easier for $\hat{y}$ either to find a new mate or be enumerated in $\hat{U}_n$ where it can rejoin its old mate x.

In [33] an element $x \in M_s$ never left M. Now x can be enumerated in A $(= U_1)$ and leave M forever. This necessitates a new kind of exclusion from $\mathcal{M}$ (Condition 3 exclusion) which causes major revisions and rearrangements in the old machinery. This extra exclusion, together with the test markers and oracle function h for $\bar{A}$, guarantee that the potential mates x do not disappear from pocket P too often.

Now suppose $\hat{y} \in \hat{Q}$ has been enumerated in V_n. To be sure that $\hat{y}$ sees some x on track D such that $x \in \hat{V}_n$ we arrange gate G_1 and Rule R_3 to accomplish the appropriate enumeration into $\hat{V}_n$ of elements x at G_1 so that $\mathcal{S}(D)$ will τ-exactly cover $\mathcal{S}(\hat{Q})$. More generally, suppose there are infinitely many members $\hat{y}$ of pocket $\hat{Q}$ in state $\nu = \langle e, \sigma, \tau\rangle$. Suppose that $\mathcal{S}(C)$ covers ν, namely $\nu' \in \mathcal{S}(C)$ i.o. for some $\nu' \geq \nu$. Let $\nu' = \langle e, \sigma', \tau'\rangle$, so that $\sigma \subseteq \sigma'$ and $\tau \supseteq \tau'$. Then it is possible to match those elements $\hat{y}$ in state ν with those elements x in state ν' by raising the state of x to $\langle e, \sigma', \tau\rangle$ (specifically by enumerating x in certain sets $\hat{V}_n$ under Rule R_3 when x passes gate G_1) and by raising the state of $\hat{y}$ to $\langle e, \sigma', \tau\rangle$ (specifically by enumerating $\hat{y}$ in certain sets $\hat{U}_n$ under Rule R_4 when $\hat{y}$ lies in pocket $\hat{Q}$.

4.3. *The construction*

At the beginning of a stage, there is at most one ball x $(\hat{x})$ on the surface of either machine M or $\hat{M}$. If there is such a ball, the stage consists of moving the ball according to the rules of Section 5 down one *track*, a section of the surface between any two of the following: gate, doors, joins, or pockets. This may cause us to enumerate x in $\hat{V}_n$ ($\hat{x}$ in $\hat{U}_n$). If there is no ball on the surface of the machine, we choose one ball above a hole and move it down the surface of the machine in successive states. If at the beginning of a stage all balls are in pockets, we enumerate another element according to g. This may cause us to place some balls above holes.

As in Section 2 we have markers $\{\Gamma_j\}_{j\in\omega}$, and the recursive function h for $\bar{A}$ satisfying (2.1) and (2.2). When an element x enters pocket P we *assign* (according to Rule R_2 of Section 5.4) a certain marker Γ_j to x. That marker remains assigned to x either forever or until x leaves pocket P, at which time that assignment is *cancelled*. This cancellation will result in an *injury* to the marker (under Definition 5.4) unless roughly x is enumerated almost immediately into U_1 $(= A)$. As in Section 2 we have a set $Z_{j,s}$ which (except for some elements in A) is the set of positions of marker Γ_j during stage s. As in Section 2 define a recursive function $\theta(j, s)$ by

$$W_{\theta(j,s)} = \bigcup \{Z_{j,t} : t \geq v\},$$

where v is the last state $\leq s$ at which Γ_j was *injured* according to Definition 5.4 in Section 5 below.

Construction.

Stage $s=0$. Do nothing.

Stage $s+1$. This stage consists of three steps.

Step 1. Adopt the first case which holds.

Case 1. Some x ($\hat{x}$) is on the surface of M ($\hat{M}$). There will be at most one such x or $\hat{x}$. Exactly one of the rules of Section 5 will apply to x ($\hat{x}$). Move x ($\hat{x}$) according to that rule.

Case 2. Some x is above a hole. If some x was placed above hole H_2 by an application of Rule R_{10}(a) at stage s, then let $x'=x$. Otherwise let x' be the least x above a hole of M.

(a) If x' is above hole H_2, see whether x' would enter pocket P if it were now placed on track C and were processed according to the rules of Section 5. If not, go to (b) of case 2. If so, then upon entering pocket P, x' would be assigned by Rule R_2 some marker Γ_j which is now unassigned. Enumerate x' in $Z_{j,s+1}$ and let t' be the least $t \geq s$ such that either

$$x' \in W_{1,t} \tag{4.1}$$

or

$$h(\theta(j,s),t)=1. \tag{4.2}$$

If (4.1), enumerate x' in U_1, and remove x' for ever from M. Otherwise go to (b) of case 2.

(b) Place x' on the surface of machine M at the end of the next track downward from the hole where x' lies.

Case 3. Some $\hat{x}$ is above a hole of $\hat{M}$. Choose the least such $\hat{x}$ and place $\hat{x}$ on the surface of $\hat{M}$ at the end of the next track downward from the hole where $\hat{x}$ lies.

Case 4. Each x ($\hat{x}$) in M ($\hat{M}$) at stage s is in a pocket. Enumerate one more value of the simultaneous enumeration g and adopt the corresponding subcase below.

(a) g enumerates $x \in W_0$ ($x \in \hat{W}_0$). Place a ball marked x ($\hat{x}$) above Hole H_1 ($\hat{H}_1$) unless x is in U_1. Enumerate $x \in U_0$ (V_0).

(b) g enumerates x in W_1 ($=A$). Enumerate x in U_1 and remove x from M forever. (If x is in pocket P, this motion of x may cause other elements to be removed from pocket P according to Rule R_{10}(c).)

(c) g enumerates $x \in W_e$ ($x \in \hat{W}_e$) for $e>1$. By the convention on the enumeration of W_0 ($\hat{W}_0$) and subcase 4(a), x ($\hat{x}$) must be in some pocket of M ($\hat{M}$) unless $x \in U_{1,s}$. If $x \in U_{1,s}$ go to stage $s+2$. Otherwise, enumerate x ($\hat{x}$) in U_e (V_e) unless x ($\hat{x}$) is in pocket P ($\hat{P}$). If x ($\hat{x}$) is in pocket P ($\hat{P}$), Rule R_{10} ($\hat{R}_{10}$) will apply to x ($\hat{x}$). If x ($\hat{x}$) is in pocket Q ($\hat{Q}$), Rule R_8 ($\hat{R}_8$) may apply to x ($\hat{x}$) and should be followed.

Step 2. Apply Rule R_4 to every element x in pocket Q in increasing order of x, and apply Rule $\hat{R}_4$ to every element $\hat{x}$ in $\hat{Q}$ in increasing order of x.

Step 3. Apply Rule R_{11} ($\hat{R}_{11}$) to every element in pocket Q ($\hat{Q}$).

This completes the construction.

Note that because of the enumeration of y' in $Z_{j,s+1}$ under case 2(a), $Z_{j,s+1}$ may contain more elements than just positions of marker Γ_j during stage $s+1$, but any such additional element is enumerated in U_1 $(=A)$ and thus has no effect on the rest of the construction. This peculiarity of case 2(a) is necessary for the following reason. Intuitively we would like $Z_{j,s+1}$ to consist exactly of the positions of marker Γ_j during stage $s+1$ as before. For other technical reasons we can only assign Γ_j to an element x' when x actually enters pocket P. But for x' above hole H_2 we would have to allow x' to pass intermediate tracks X, thereby adding its current full c-states to $\mathscr{S}(X)$, before x' is enumerated in $Z_{j,s+1}$ and tested by (4.1) and (4.2) for membership in $\bar{A}$. If (4.1) holds after this testing (so x' enters $U_1 = A$ and is removed from M forever) we do not want such a 'false' x' to have added any ν to any $\mathscr{S}(X)$.

5. The rules

5.1. Preliminaries

We now give the rules and some of the lemmas which the rules are designed to insure. Whenever possible these lemmas are placed just after the rules which yield them in order to illuminate the action of each rule. Other lemmas will be deferred until Section 6. The rules with numbered subscripts correspond to those of [33] although they are sometimes substantially modified. In addition, the following rule summarizes the motion of x when x is enumerated in U_1 $(=A)$.

Rule $\mathbf{R_A}$. If $x \in U_{1,s+1} - U_{1,s}$ (necessarily because either case 2(a) or case 4(b) of the construction applies to x at stage $s+1$) then x is permanently removed from M.

The rules and lemmas for the dual part, matching $\hat{Q}$ to P, are similar but not identical to those in [33]. They will be stated here for completeness, but proofs of any lemmas will be omitted whenever they are essentially the same as in [33].

Rule R_1 $(\hat{R}_1)$ determines which balls enter tracks C_1 and C_2 $(\hat{C}_1$ and $\hat{C}_2)$ when they reach door D_1. We let $\mathscr{R}_s$ denote a certain sequence defined by induction on s and containing (exactly once) each pair $\langle \nu, j\rangle$ for all $j \in \{1, 2\}$ and all full e-states ν, for $e < s$. Let $\mathscr{R}_0 = \{\langle \nu_{-1}, 1\rangle, \langle \nu_{-1}, 2\rangle\}$.

Rule $\mathbf{R_1}$. Suppose that sequence $\mathscr{R}_s$ is given. If an element x enters track C at stage s, then at stage $s+1$ it enters either tack C_1 or C_2 (with $\nu(s+1, x, x) = \nu(s, x, x)$) as follows. Let $\langle \nu', i'\rangle$ be the first pair $\langle \nu, i\rangle$ on the sequence $\mathscr{R}_s$ such that $\nu \leqslant \nu(s, x, x)$. Remove $\langle \nu', i'\rangle$ from its present position on $\mathscr{R}_s$, place it at the end of the sequence, and place x on track $C_{i'}$. In this case we say that $\langle \nu', i'\rangle$ is *reset* at stage $s+1$. Finally, whether an element x entered track C or not, add $\langle \nu, i\rangle$ at the end of the sequence (in any fixed effective order) for each $i \in \{1, 2\}$ and each full s-state ν. Let $\mathscr{R}_{s+1}$ denote the resulting sequence.

Rule $\hat{R}_1$. Like R_1 but with $\hat{C}$, $\hat{C}_1$, $\hat{C}_2$ in place of C, C_1, C_2.

Lemma 5.1. *Streams C_1 and C_2 ($\hat{C}_1$ and $\hat{C}_2$) are each equivalent to stream C ($\hat{C}$) (namely $\mathcal{S}(C_1)$ exactly covers $\mathcal{S}(C_2)$ and vice versa).*

Proof. See [33, Lemma 4.1].

5.2. The easy rules and the lemmas on motion

Rule R_2 will be precisely stated in Section 5.4. Briefly, if x enters track D at stage s, then at stage $s+1$, it enters either pocket P or track D_1 according to which elements are in P_s and $\hat{Q}_{s+1}$. (Very roughly, x enters pocket P just if some $\hat{y} \in \hat{Q}_{s+1}$ needs x as a mate.) If $x \in P_{s+1} - P_s$, there may be several $y \in P_s - P_{s+1}$. These are handled by Rule R_{12}. Rule $\hat{R}_2$ is similar.

In Sections 5.6 and 5.7, we shall give Rule R_3 which governs the enumeration of x in $\hat{V}_{n,s+1}$ as x passes gage G_1, and Rule R_4 whereby $x \in Q_s \cap Q_{s+1}$ may sometimes be enumerated in $\hat{V}_{n,s+1}$. Rules $\hat{R}_3$ and $\hat{R}_4$ for $\hat{U}_n$ are similar. There are no rules R_5 or $\hat{R}_5$, although we have preserved the numbering of rules to agree with that in [33].

Rule R_6. Element $x \in \hat{V}_{n,s+1} - \hat{V}_{n,s}$ only if at stage $s+1$, Rule R_3 or R_4 applies to x.

Rule R_7. If $x \in U_{n,s+1} - U_{n,s}$ for $n > 1$, then x is in some pocket of M at the end of stage s.

(Rule R_7 is simply a remark which follows from step 1 case 4(c) of the construction.) In particular, Rules R_6 and R_7 imply that the only change in full x-state of x allowed during a play (until x reaches a pocket) is under R_3 or R_4. Rules $\hat{R}_6$ and $\hat{R}_7$ for $\hat{M}$ are similar. In addition to the above rules for a play, we need rules concerning the pockets.

Rule R_8. If $x \in Q_s$, and if $x \in U_{n,s+1} - U_{n,s}$ for some $n \leq X$, remove x from pocket Q at stage $s+1$, and place x above hole H_2.

Rule $\hat{R}_8$. If $\hat{x} \in \hat{Q}_s$, and if $x \in V_{n,s+1} - V_{n,s}$ for some $n \leq x$, remove $\hat{x}$ from pocket $\hat{Q}$ at stage $s+1$, and place $\hat{x}$ above hold $\hat{H}_2$.

Rule R_9. If $x \in Q_s$, then $x \in Q_{s+1}$, unless x is removed at stage $s+1$ under R_8 or Rule R_A.

In Section 5.4 we shall give Rules R_{10} and R_{11} which tell when $x \in P_s$ may be removed from pocket P at stage $s+1$ in case $x \in U_{n,s+1} - U_{n,s}$ or $\hat{Q}_s \neq \hat{Q}_{s+1}$. Rules

$\hat{R}_9, \hat{R}_{10}, \hat{R}_{11}$, for pockets $\hat{Q}$ and $\hat{P}$ are similar, except that in Rule $\hat{R}_9$ there is no reference to Rule R_A.

Rule R_{12}. If $x \in P_s$ then $x \in P_{s+1}$ unless x is removed from P at stage $s+1$ by Rule R_A, R_2, R_{10}, or R_{11}. If x is removed at stage $s+1$ by R_2, R_{10}, or R_{11} and x last entered P at stage $t \leq s$, then at stage $s+1$, x is placed above hole H_3 if $\nu(s+1, x, x) = \nu(t, x, x)$, and above hole H_2 otherwise. When x is removed from pocket P, the assignment of its current marker Γ_j is cancelled.

Rule $\hat{R}_{12}$ is similar except that references to R_A and Γ_j are omitted.

Rule R_{13}. If at the end of stage s, x is on any of the tracks $C_0, C_2, C_3, C_4, C_5, D_1$, place x at stage $s+1$ on the next track or pocket in the downward direction (the direction of the arrows).

Lemma 5.2. *Stream C σ-exactly covers any stream X of M and $\hat{C}$ τ-exactly dual covers any stream $\hat{X}$ of $\hat{M}$.*

Proof. By Rules R_5 and R_7, the only change in full e-state of x during a play (as defined in Section 4.2) is under R_3 or R_4, in which case the enumeration is only in $\hat{V}_n$ for certain n. According to case 4 of the construction, the only enumeration of x in U_n while x is in M occurs when x is in a pocket. If this pocket is P, then by Rule R_{12}, x must go to hole H_2 when it leaves P. Thus, any element x on a track X of M at stage s must have been on track C at some stage $t \leq s$ such that $\sigma(s, x, x) = \sigma(t, x, x)$ and $\tau(t, x, x) \subseteq \tau(s, x, x)$, namely, $\nu(s, x, x) \leq^\sigma \nu(t, x, x)$. (The only exception is track $X = C_0$ in which case x was on track C_0 at stage $t-1$ with $\nu(t-1, x, x) = \nu(t, x, x)$.) The case of $\hat{C}$ and $\hat{X}$ is entirely dual.

Lemma 5.3. *Each element x ($\hat{x}$) re-enters the surface of M ($\hat{M}$) at most finitely often.*

Proof. Consider an element x of M. (This case of $\hat{x} \in \hat{M}$ is similar.) If x enters U_1 ($= A$), x is removed from M under R_A and x never returns. Otherwise, x is eventually placed above, and enters the surface of M from, hole H_1 at most once (during step 1 of case 4(a) of the construction). Suppose x is placed above hole H_2 at stage $s+1$. Then $x \in P_s$ or Q_s. If $x \in Q_s - Q_{s+1}$, then $\nu(s+1, x, x) \neq \nu(s, x, x)$ by Rule R_9, which can happen only finitely often. Similarly, if $x \in P_s - P_{s+1}$, $\nu(s+1, x, x) \neq \nu(t, x, x)$ by Rule R_{12} where t is the last stage before s that x entered P. Thus x can be placed above hole H_2 only finitely often. If x is placed above hole H_3, x can only re-enter M from pocket Q and hence via hole H_2.

Lemma 5.4. *Each element x ($\hat{x}$) is in some pocket for cofinitely many stages or $x \in U_1$ ($= A$).*

Proof. By Lemma 5.3, each element x re-enters only finitely often. Once x is on the surface of M it will reach a pocket by the construction. The only possible obstacle then is for x to remain above a hole cofinitely often. Choose any stage s such that x is above a hole. At stage s, only finitely many elements are in M_s. By Lemma 5.3, each of the elements besides x can generate only finitely many moves; eventually x will be the x of case 2 of the construction and will re-enter the surface of M, or x will be enumerated in U_1.

5.3. The sequence $\mathcal{K}_s$ and the row function $r(s, \nu)$

In Definition 5.1, we define uniformly recursively in s a sequence $\mathcal{K}_s$ of full states which contains each full j-state, $j < s$, exactly once. (This is the same definition as in [33].)

Definition 5.1. (a) If $\nu_1, \nu_2 \in \mathcal{K}_s$, we write $\nu_1 \leq_s^{\#} \nu_2$ if ν_1 precedes ν_2 in the sequence $\mathcal{K}_s$.

(b) Let $\mathcal{K}_0 = \{\nu_{-1}\}$. Given $\mathcal{K}_s$, define

$$\mathcal{K}_{s+1}^2 = \{\nu \colon \nu \in \mathcal{K}_s \text{ and } (\exists \nu')[\nu <^{\tau} \nu' \text{ and } [\nu' \in \mathcal{S}_{s+1}(C) \text{ or } \nu' \in \mathcal{S}_{s+1}(D)]]\}.$$

Define $\mathcal{K}_{s+1}^1 = \mathcal{K}_s - \mathcal{K}_{s+1}^2$. Let $\mathcal{K}_{s+1}^3$ be the sequence of all full s-states arranged in some effective order (uniformly in s) such that $\nu' = \langle s, \sigma', \tau' \rangle$ precedes $\nu = \langle s, \sigma, \tau \rangle$ if $\tau' \subset \tau$ or if $\tau = \tau'$ and $\sigma' \supseteq \sigma$. Let $\mathcal{K}_{s+1}$ denote the sequence which is the concatenation of the sequences $\mathcal{K}_{s+1}^1, \mathcal{K}_{s+1}^2$, and $\mathcal{K}_{s+1}^3$ in that order, where $\mathcal{K}_{s+1}^1, \mathcal{K}_{s+1}^2$ have the orderings induced by $\leq_s^{\#}$.

(c) Define a recursive function (*row function*) $r(s, \nu)$ as follows:

$$\begin{aligned}
&r(0, \nu_{-1}) = 1, \qquad r(0, \nu) \text{ is undefined if } \nu \neq \nu_{-1};\\
&r(s+1, \nu) = r(s, \nu) \quad \text{if } \nu \in \mathcal{K}_{s+1}^1;\\
&r(s+1, \nu) = \max\{r(s, \nu') \colon r(s, \nu') \text{ is defined}\} + k\\
&\qquad\qquad \text{if, } \nu \in \mathcal{K}_{s+1}^2 \cup \mathcal{K}_{s+1}^3 \text{ and is the } k\text{th such in}\\
&\qquad\qquad \text{the ordering } \leq_{s+1}^{\#};\\
&r(s+1, \nu) \text{ is undefined if } \nu \notin \mathcal{K}_{s+1}.
\end{aligned}$$

(d) Let $\mathcal{K}_\omega$ denote those ν which are in $\mathcal{K}_{s+1}^2$, for only finitely many s.

(e) ν^k is the unique ν such that $(\exists s)[r(s, \nu) = k]$.

Think of the list $\mathcal{K}_s$ as a list of full j-states, $j < s$, each occupying a row. At stage $s+1$, certain full j-states (those in $\mathcal{K}_{s+1}^2$) are *reset*; i.e., they are removed from the list and placed on 'fresh' rows below the list $\mathcal{K}_s$. Also at stage $s+1$, the full s-states are added to $\mathcal{K}_{s+1}$ in fresh rows below $\mathcal{K}_{s+1}^2$. In this picture, $r(s, \nu)$ is the number of the row which ν occupies in $\mathcal{K}_s$. Note that $\nu \in \mathcal{K}_\omega$ iff $\lim_s r(s, \nu)$ exists. Also note that $r(s, \nu) \leq r(s, \nu')$ iff $\nu \leq_s^{\#} \nu'$. Each row k will eventually be occupied by some ν, which we deonte by ν^k. This ν will either remain in row k forever or move to a new row in which case no other $\nu' \neq \nu^k$ ever occupies row k.

If $\mathcal{S}$ is a sequence of full e-states, we say ν_0 is *maximal with respect to* $\mathcal{S}$ if

$(\forall\nu)[[\nu \in \mathcal{S}$ i.o. and $\nu_0 \leqslant^{\tau} \nu] \Rightarrow \nu = \nu_0]$.

In [33] we easily proved the following lemmas:

Lemma 5.5 ([33, Lemma 5.1). *For all e-states* σ, σ', τ *and all* $s > e$,

$$\sigma' \supseteq \sigma \Rightarrow [\langle E, \sigma', \tau\rangle \leqslant_s^{\#} \langle e, \sigma, \tau\rangle].$$

Lemma 5.6 ([33], Lemma 5.2). *For all* ν, $\nu \in \mathcal{K}_\omega$ *iff* ν *is maximal with respect to* $\mathcal{S}(D)$.

5.4. The functions $\hat{q}(s, \nu), p(s, \nu)$; *the Rules* R_2, R_{10}, R_{11} *and the markers* $\Gamma_{\langle k,l,\nu\rangle}$

Let P_s, Q_s, M_s $(\hat{P}_s, \hat{Q}_s, \hat{M}_s)$ denote the set of elements in pocket P, pocket Q, machine M $(\hat{P}, \hat{Q}, \hat{M})$ at the end of stage s.

Definition 5.2. Define (in increasing order of $<$) a function $\hat{q}(s, \nu)$ by $\hat{q}(s, \nu) = \mu\hat{y} \in \hat{Q}_s$ such that

(1) $\nu \leqslant \nu(s, y, \hat{y})$,

(2) $\hat{q}(s, \nu') \neq \hat{y}$ for any $\nu' < \nu$.

If $\hat{y}$ does not exist, then $\hat{q}(s, \nu)$ is undefined.

Clearly, if $\hat{y} \in \hat{Q}_s$, $\hat{y} = \hat{q}(s, \nu)$ for some ν. Roughly, the goal of Rule R_2 is to define a function $p(s, \nu)$ with values in P_s so that if $\hat{q}(\nu) = \lim_s \hat{q}(s, \nu)$ and $p(\nu) = \lim_s p(s, \nu)$, then $\hat{q}(\nu) \to p(\nu)$ is the intended piece of the function ψ^{-1} mapping $\hat{Q}$ to P. The major difficulty is that $p(\nu)$ will not necessarily exist even if $\hat{q}(\nu)$ does. In this case, we will try to arrange that $\hat{q}(\nu')$ exists for only finitely many $\nu' > \nu$, thereby enabling us to construct at least a finite-to-one map. Our major concern is to control the cases in which $p(s, \nu)$ becomes undefined infinitely often for a fixed ν.

Definition 5.3. If $x \in M_s$ and ν is a full e-state such that $\nu \leqslant \nu(s, x, x)$, the *$\nu$-rank of x at stage s*, denoted $\rho(s, \nu, x)$, is the minimum of the set $\{r(s, \nu') : \nu \leqslant \nu' \leqslant \nu(s, x, x)\}$.

Rule R_2, which governs the selection of $p(s, \nu)$, will depend on the function ρ defined above as well as certain markers $\Gamma_{\langle k,l,\nu\rangle}$. Rule R_2 will choose $p(s, \nu)$ so that the value $\rho(s, \nu, p(s, \nu))$ is as small as possible. (The reason for this measure of 'desirability' of candidates for $p(s, \nu)$ is not obvious but will come out in the proof of Lemma 6.1.) The marker $\Gamma_{\langle k,l,\nu\rangle}$ will be used to prevent $p(s, \nu)$ from being enumerated into A infinitely often where

$$k = \rho(s, \nu, p(s, \nu)) \quad \text{and} \quad l = (\mu t)(\forall \nu' \leqslant \nu)[\hat{q}(l, \nu') = \hat{q}(\nu')].$$

Recall that ν^k is the unique ν such that $(\exists s)[r(s,\nu)=k]$. For each $k, l \in N$, form triples $\langle k, l, \nu\rangle$ where $\nu \leqslant \nu^k$. Linearly order these triples by

$$\langle k, l, \nu\rangle < \langle k', l', \nu'\rangle \quad \text{if } 2^k 3^l < 2^{k'} 3^{l'}$$

or

$$k=k', \qquad l=l' \quad \text{and} \quad \nu \prec \nu'.$$

We suppose that these triples have been put in 1–1 order preserving correspondence with N and we denote the integer corresponding to $\langle k, l, \nu\rangle$ also by $\langle k, l, \nu\rangle$. When an element x enters pocket P (necessarily by Rule R_2) we *assign* a certain marker $\Gamma_{\langle k,l,\nu\rangle}$ to it. That marker remains *assigned* to x until x leaves pocket P, at which time the assignment is *cancelled*.

Rule R_2. Suppose that element x enters track D at stage s. Let ν' be the first ν (in the ordering $\prec$) such that:

(i) $\nu \leqslant \nu(s, x, x)$;

(ii) $\hat{q}(s,\nu)$ is defined; and

(iii) either $p(s,\nu)$ is undefined, or

$$\rho(s,\nu,x) < \rho(s,\nu,p(s,\nu)).$$

If ν' fails to exist, place x on track D_t at stage $s+1$. If ν' exists, then at stage $s+1$:

(a) define $p(s+1,\nu')=x$, place x in pocket P;

(b) remove from P all elements $p(s,\nu'')$ such that $\nu' \leqslant \nu''$;

(c) assign the marker $\Gamma_{\langle k,l,\nu'\rangle}$ to x where

$$k=\rho(s,\nu',x) \quad \text{and} \quad l=(\mu t)(\forall z)_{t\leqslant z\leqslant s}(\forall\, \nu \leqslant \nu')[\hat{q}(s,\nu)=\hat{q}(z,\nu)].$$

Rule R_{10}. (a) If $x=p(s,\nu)\in P_s$ for some ν such that $|\nu|=e$, and if x is enumerated in W_n at stage $s+1$ for some $n, 1<n\leqslant e$, but $x\notin U_{n,s}$, then at stage $s+1$ enumerate x in U_n and remove x from pocket P.

(b) If the above hypotheses hold except that $n>e>1$, then leave x in pocket P and go to stage $s+2$.

(c) If $x=p(s,\nu)\in P_s$ and x is removed from P at stage $s+1$ under either Rule R_{10}(a) or R_A, then all elements $p(s,\nu')$, $\nu\prec\nu'$, are also removed from pocket P at stage $s+1$.

Rule R_{11}. If $\hat{q}(s,\nu)\neq\hat{q}(s+1,\nu)$ or $\hat{q}(s+1,\nu)$ is undefined, then remove from pocket P at stage $s+1$ all $p(s,\nu')$ such that $\nu\leqslant\nu'$.

Note that by Rule R_{12}, x is removed from pocket P only if Rule R_A, R_2, R_{10} or R_{11} applies to x, in which case x is moved according to Rule R_{12}. Rules $\hat{R}_2$, $\hat{R}_{10}$, $\hat{R}_{11}$, the pockets $\hat{P}$ and Q, and the functions $\hat{p}$ and q are duals to the above except that the markers $\Gamma_{\langle k,l,\nu\rangle}$ and references to Rule R_A are omitted.

Note that these rules guarantee:

$$p(s, \nu) \text{ defined} \Rightarrow \hat{q}(s, \nu) \text{ defined}, \tag{5.1}$$

$$p(s, \nu) \text{ defined} \Rightarrow (\forall \nu')[\nu' < \nu \Rightarrow p(s, \nu') \text{ defined}], \tag{5.2}$$

and

$$[p(s+1, \nu) \neq p(s, \nu) \text{ or } \hat{q}(s+1, \nu) \neq \hat{q}(s, \nu)] \\ \Rightarrow (\forall \nu')[\nu \leqslant \nu' \Rightarrow p(s, \nu') \notin P_{s+1}]. \tag{5.3}$$

Note that by Rule R_2 some marker $\Gamma_{\langle k,l,\nu\rangle}$ is assigned to an element x precisely when x enters pocket P. That assignment remains unchanged while x remains in P, but the assignment is cancelled by Rule R_{12} when (if ever) x leaves P, say at stage $s+1$. This cancellation results in an *injury* (in the sense of Section 2) to $\Gamma_{\langle k,l,\nu\rangle}$ unless x is enumerated in U_1 $(=A)$ by the end of stage $s+2$. The point is that if x is removed from P at stage $s+1$ by Rule R_{10}(a), then by case 2 of the construction, x is immediately processed at stage $s+2$ and is either enumerated in U_1 or drops to the surface of M. This plays a crucial role in the proof later of Lemma 6.1.

Definition 5.4. If the assignment of marker Γ_j to x is cancelled at stage $s+1$ (so $x \in P_s - P_{s+1}$), then marker Γ_j is *injured at stage* $s+1$ unless $x \in U_{1,s+2}$.

Note that if x enters P at stage $s+1$, say $x = p(s+1, \nu)$, then x is assigned by Rule R_2(c) some marker $\Gamma_{\langle k,l,\nu\rangle}$ where $k = \rho(s+1, \nu, x)$. While x remains in P at stages $t > s$, $\rho(t, \nu, x)$ could change if either (1) $x \in U_{n,t} - U_{n,t-1}$ for some $n > e = |\nu|$; or (2) ν^k is reset on $\mathcal{K}$. Rule R_{10}(b) prevents changes of type (1), and changes of type (2) will not bother us in Lemmas 6.1 and 6.2 which are our main concern, because there we can fix j and choose t such that for all ν', if $r(t, \nu') \leqslant j$, then $r(t, \nu') = \lim_s r(t, s)$. Nevertheless, since in general we cannot prevent changes of type (2), the reader should note that x in P at stage t and associated with marker $\Gamma_{\langle k,l,\nu\rangle}$ may have $\rho(t, \nu, x) = k' \neq k$.

5.5. The sequences $\mathcal{M}, \mathcal{P}$ and the functions $d(s, \hat{x})$ and $\nu^*(s, \hat{x})$

By induction on s, we now define recursively, uniformly in s, sequences of full e-states $\mathcal{M}_s, \mathcal{P}_s$ such that $\mathcal{M}_s \subseteq \mathcal{P}_s \subseteq \mathcal{K}_s$ for all s. Our main goal is to prove that D exactly covers $\hat{Q}$ so that $\hat{Q}$ sees enough 'mates' to achieve the desired piece of the permutation. (Here $\hat{Q}$ can be considered a stream in the following sense: let $\mathcal{S}_{s+1}(\hat{Q}) = \{\nu(s+1, e, \hat{x}): e \leqslant x$ and either $\hat{x} \in \hat{Q}_{s+1} - \hat{Q}_s$ or $\hat{x} \in \hat{Q}_{s+1} \cap \hat{Q}_s$ and $\nu(s+1, x, \hat{x}) \neq \nu(s, x, \hat{x})\}$.) We do this by insuring that C covers *every* stream $\hat{X}$ of $\hat{M}$ and that D τ-exactly covers every stream that C covers.

Definition 5.5. (a) Let $\mathcal{M}_0 = \{\nu_{-1}\}$. Given $\mathcal{K}_{s+1}$ and $\mathcal{M}_s$ we define $\mathcal{M}_{s+1}$ as follows. Let $\nu = \langle e, \sigma, \tau\rangle$ and suppose we have already determined whether $\nu' \in \mathcal{M}_{s+1}$ if

$|\nu'|<e$. We say ν *is excluded from* $\mathcal{M}_{s+1}$ if one of the following conditions holds:

Condition 1. $(\exists\nu')[r(s,\nu')\le r(s,\nu)$ and $\nu'\in\mathcal{K}^{2}_{s+1}]$.

Condition 2. $(\exists\nu')(\exists\hat{X})[\nu'\in\mathcal{S}_s(\hat{X})-\mathcal{P}_s$ and $|\nu'|<e]$, where $\hat{X}$ is any stream of $\hat{M}^{\tau}$ including $\hat{Q}$.

Condition 3. There is a full j-state ν' and an element x such that

(i) $x=p(s,\nu')$,

(ii) $x\in P_s-P_{s+1}$ because either Rule R_{10}(a) or R_A applied to x at stage $s+1$,

(iii) $x\in U_{1,s+2}$,

(iv) $r(s,\nu)\ge\max\{k,l\}$ where $k=\rho(s,\nu',x)$, and

$$l=(\mu t\le s)(\forall z)_{t\le z\le s}(\forall\nu''\preccurlyeq\nu')[\hat{q}(z,\nu'')=\hat{q}(s,\nu'')].$$

Define $\nu\in\mathcal{M}_{s+1}$ if ν is not excluded from $\mathcal{M}_{s+1}$ and either $\nu\in\mathcal{M}_s$ or $\nu\in\mathcal{S}_{s+1}(D)$.

(b) Let $\mathcal{P}_{s+1}=\{\nu:(\exists\nu')[\nu'\in\mathcal{M}_{s+1}$ and $\nu\le^{\tau}\nu']\}$.

(c) Let $\mathcal{M}_\omega=\{\nu:\nu\in\mathcal{M}_s$ for almost all $s\}$.

(d) Let $\mathcal{P}_\omega=\{\nu:\nu\in\mathcal{P}_s$ for almost all $s\}$.

(e) Let $v_{\hat{x}}=\mu t[\hat{x}\in\hat{M}_t]$. Define

$$d(s,\hat{x})=\max\{e:\nu(s,e,\hat{x})\in\mathcal{P}_s\text{ and }(\forall t)[v_{\hat{x}}\le t<s\Rightarrow e\le d(t,\hat{x})]\}.$$

For other $\hat{x}$ and s let $d(s,\hat{x})=-1$. (Note that $\lim_s d(s,\hat{x})$ exists.)

(f) Let $\nu^*(s,\hat{x})=[(\nu(s,x,\hat{x})]_{d(s,\hat{x})}$ and $\nu^*(\hat{x})=\lim_s\nu^*(s,\hat{x})$.

Sequences $\mathcal{K}'_s,\mathcal{M}'_s,\mathcal{P}'_s$ and functions $d'(s,x),\nu^*(s,x)$ for $x\in M_s$ are defined similarly to $\mathcal{K}_s,\mathcal{M}_s,\mathcal{P}_s$, $d(s,\hat{x})$ and $\nu^*(s,x)$ with $\mathcal{S}(C)$, $\mathcal{S}(D)$, and $\mathcal{S}(X)$ replaced by $\mathcal{S}(\hat{C})$, $\mathcal{S}(\hat{D})$, and $\mathcal{S}(\hat{X})$ respectively, and with the roles of σ and τ interchanged except that there is no Condition 3 exclusion from $\mathcal{M}'_{s+1}$. All the obvious dual lemmas hold. Note that the functions d and d' clearly satisfy the following:

$$(\forall x)(\exists s)(\forall t)_{\ge s}[d(t+1,\hat{x})\le d(t,\hat{x})],\tag{5.4}$$

$$(\forall x)(\exists s)(\forall t)_{\ge s}[d'(t+1,x)\le d'(t,x)],\tag{5.5}$$

and hence

$$(\forall x)[\lim_s d(s,\hat{x})\text{ and }\lim_s d'(s,x)\text{ exist}].\tag{5.6}$$

Lemma 5.7. *Suppose that each* $\nu\in\mathcal{K}_\omega$ *is excluded from* $\mathcal{M}_{s+1}$ *under Condition* 2 *or* 3 *of Definition* 5.5(a) *for at most finitely many s. Then*

(i) $\nu\in\mathcal{K}_\omega$ *and* $\nu\in\mathcal{S}(D)$ *i.o.* $\Rightarrow\nu\in\mathcal{M}_\omega$,

(ii) $\nu\in\mathcal{P}_s$ *for infinitely many* $s\Rightarrow\nu\in\mathcal{P}_\omega$.

Proof. (i) Fix $\nu\in\mathcal{K}_\omega$. Choose t such that $r(t,\nu')=\lim_s r(s,\nu')$ for all ν' such that $r(t,\nu')\le r(t,\nu)$. Let $j=r(t,\nu)$. Now ν is never excluded from $\mathcal{M}_{s+1}$ by Condition 1 at any stage $s+1>t$. Hence, there is some $u\ge t$ such that ν is never excluded from $\mathcal{M}_{s+1}$ if $s\ge u$. Thus $\nu\in\mathcal{M}_\omega$ because $\nu\in\mathcal{S}(D)$ i.o.

For (ii), fix $\nu_1 \in \mathscr{P}_s$ for infinitely many s. Then either $\nu_1 \in \mathscr{P}_\omega$ immediately or else $\nu_1 \in \mathscr{P}_{s+1} - \mathscr{P}_s$ for infinitely many s. For such an s, some $\nu_2 \in \mathscr{M}_{s+1} - \mathscr{M}_s$, and hence $\nu_2 \in \mathscr{S}_{s+1}(D)$, where $\nu_1 \leqslant^\tau \nu_2$. Choose any $\nu_3 \in \mathscr{S}(D)$ i.o. where $\nu_1 \leqslant^\tau \nu_3$, and ν_3 is maximal with respect to $\mathscr{S}(D)$. By part (i), $\nu_3 \in \mathscr{M}_\omega$ and hence $\nu_1 \in \mathscr{P}_\omega$, thereby proving Lemma 5.7.

We now give some intuition as to these lists. First, $\mathscr{K}_\omega$ is simply the maximal full e-states of $\mathscr{S}(D)$ with respect to τ-exact covering. $\mathscr{M}_\omega$, except for the complications of Conditions (2) and (3) of Definition 5.5(a), is an approximation to those states in $\mathscr{K}_\omega$ which appear infinitely often on $\mathscr{S}(D)$. Thus $\mathscr{P}_\omega$ is a recursive approximation to those full e-states that D τ-exactly covers. If we knew the full e-states that D does in fact τ-exactly cover, we could guarantee that D τ-exactly covered $\hat{Q}_\omega$ by only enumerating elements of $\hat{M}$ into these states. As it is, $\nu^*(s, \hat{x})$ is simply a 'guess' as to the largest initial segment of $\nu(s, x, \hat{x})$ which D τ-exactly covers.

Condition 2 in Definition 5.5(a) is a technical device necessary to prove Lemma 5.14. Condition 3, on the other hand, is a crucial difference between this construction and [33]. It asserts that if an element $\hat{q}(s, \nu')$ with 'priority' l has an apparent mate $x = p(s, \nu')$ of 'rank' k and x is enumerated into U_i $(= A)$, then every ν of lower priority, i.e., $r(s, \nu) \geqslant \max\{k, l\}$, is excluded from $\mathscr{M}_{s+1}$. This extra exclusion is necessary in the proof of Lemma 6.1, but Lemma 6.2 shows that any $\nu \in \mathscr{K}_\omega$ is excluded under Condition 3 at most finitely often.

5.6. *The sequence $\mathscr{H}$, gate G_1 and Rule R_3*

The next rule, Rule R_3, determines what happens when an element x arrives at gate G_1. Here the element x may be enumerated in certain sets $\hat{V}_n$. The purpose of this rule is to insure that D τ-exactly covers any stream $\hat{X}$ which C covers. In the next rule and associated lemmas, X $(\hat{X})$ ranges over any proper track of M $(\hat{M})$, i.e., not Q $(\hat{Q})$.

Rule R_3 involves a certain r.e. sequence $\mathscr{H}$ of full e-states, which is the concatenation of the finite sequences $\mathscr{H}_s$, $s \in N$, defined as follows. $\mathscr{H}_s = \emptyset$ unless there is some track $\hat{X}$ of $\hat{M}$, some element $\hat{y}_1$ on track $\hat{X}$ at stage s, and some full e_1-state ν_1 such that

$$\nu_1 \in \mathscr{S}_s(\hat{X}) - \mathscr{P}_s$$

via $\hat{y}_1$, in which case $\mathscr{H}_s$ consists of the following full e_1-states (in some effective order uniformly in s):

$$\{\nu\colon (\exists t)_{\leqslant s}[\hat{y}_1 \in \hat{M}_t \text{ and } \nu(t, e_1, \hat{y}_1) \leqslant^\tau \nu]\}.$$

Once added to $\mathscr{H}$, a given ν is never removed from $\mathscr{H}$ or altered in position, although it may later be *checked* during an application of Rule R_3. Let $\mathscr{H}_{\leqslant s}$ denote the sequence of elements added to $\mathscr{H}$ by the end of stage s.

Rule R_3. Suppose an element x enters track C_1 at stage s. Let $\nu_0 = \langle x, \sigma_0, \tau_0 \rangle$ denote $\nu(s, x, x)$. Let $\nu_1 = \langle e_1, \sigma_1, \tau_1 \rangle$ be the first member $\nu = \langle e, \sigma, \tau \rangle \in \mathscr{H}_{\leqslant s}$ such that:

(i) ν has not been checked by the end of stage s;

(ii) $e \leqslant x$; and

(iii) $\nu \leqslant^{\sigma} [\nu_0]_e$ (i.e., $\sigma = [\sigma_0]_e$, and $\tau \supseteq [\tau_0]_e$).

If ν_1 exists, then at stage $s+1$ check ν_1, enumerate x in $\hat{V}_{n,s+1}$ for each $n \leqslant e_1$ such that $n \in \tau_1 - \tau_0$ (so that $\nu(s+1, e_1, x) = \nu_1$), and place x on track C_3. If ν_1 fails to exist, then at stage $s+1$ place x on track C_4, and let $\nu(s+1, x, x) = \nu(s, x, x)$.

Rule $\hat{\mathrm{R}}_3$. Same as Rule R_3, but with $C_1, C_3, C_4, \hat{C}, V_n, \mathscr{H}$ replaced by $\hat{C}_1, \hat{C}_3, \hat{C}_4, C, \hat{U}_n$, and $\mathscr{H}'$ (which is defined in the analogous way using $\mathscr{P}'_s$ and $\mathscr{S}_s(X)$), and with the roles of σ and τ interchanged.

We shall later prove that each ν may be added to $\mathscr{H}$ at most finitely often. Thus, the following lemma will show that the enumeration under Rule R_3 above has been sufficiently restrained so that $\hat{C}$ dual covers C_3. The proofs of these lemmas are exactly the same as [33, Lemmas 5.4 and 5.5]. Let $\mathscr{S}^e(X) = \{\nu : \nu \in \mathscr{S}(X) \,\&\, |\nu| \leqslant e\}$.

Lemmas 5.8. *Fix e. If each ν such that $|\nu| < e$ is added to $\mathscr{H}$ at most finitely often, then $\hat{C}$ dual covers $\mathscr{S}^e(C_3)$.*

Lemma 5.9. *Fix e. If each ν such that $|\nu| < e$ is added to $\mathscr{H}'$ at most finitely often, then C covers $\mathscr{S}^e(\hat{C}_3)$.*

The purpose of Rule R_3 is to allow sufficient enumeration at Gate G_1 to yield:

Lemma 5.10. *If $\hat{X}$ is any stream of $\hat{M}$, and C covers $\mathscr{S}(\hat{X})$, then D τ-exactly covers $\mathscr{S}(\hat{X})$.*

Proof. Fix a stream $\hat{X}$ of $\hat{M}$, and assume for a contradiction that some $\nu_1 \in \mathscr{S}(\hat{X})$ i.o. and C covers ν_1 but no $\nu \in \mathscr{S}(D)$ i.o., for $\nu_1 \leqslant^{\tau} \nu$. Let $\nu_1 = \langle e, \sigma_1, \tau_1 \rangle$. Then there exists s' such that no ν is added to $\mathscr{S}(D)$ at any stage $s \geqslant s'$ if $\nu_1 \leqslant^{\tau} [\nu]_e$. Choose $e' \geqslant e$ and $e' > |\nu|$ for any $\nu \in \mathscr{S}(D)$ such that $\nu_1 \leqslant^{\tau} [\nu]_e$. Replacing ν_1 by some extension of length e' if necessary we may assume that $\nu_1 \in \mathscr{S}(\hat{X})$ i.o., C covers ν_1, and

$$(\forall \nu)[\nu_1 \leqslant^{\tau} [\nu]_e \Rightarrow \nu \notin \mathscr{S}(D)]. \tag{5.7}$$

By (5.7), $\nu_1 \notin \mathscr{P}_s$ for any s. Thus, by the definition of $\mathscr{H}$, each $\nu \in \mathscr{H}$ i.o. if $\nu_1 \leqslant^{\tau} \nu$.

Since C covers ν_1, some $\nu_2 = \langle e, \sigma_2, \tau_0 \rangle \in \mathscr{S}(C)$ i.o. (and hence $\nu_2 \in \mathscr{S}(C_1)$ i.o.) where $\sigma_2 \supseteq \sigma_1$ and $\tau_0 \subseteq \tau_1$. Furthermore, $\nu_1 \in \mathscr{S}_s(\hat{X}) - \mathscr{P}_s$ for infinitely many s implies that $\nu_3 = \langle e, \sigma_2, \tau_1 \rangle \in \mathscr{H}$ i.o.

Now $\nu_1 \leqslant^{\tau} \nu_3$, and thus by (5.7) $\nu_3 \notin \mathcal{S}(D)$. Hence, $\nu_3 \notin \mathcal{S}(C_3)$ and ν_3 once added to $\mathcal{H}$ is never checked under Rule R_3. Choose s_0 such that no ν preceding ν_3 on the sequence $\mathcal{H}$ is checked at any stage $s \geqslant s$. Choose $s_1 \geqslant s_0$ such that some $x \geqslant e$ enters track C_1 at stage s_1, where $\nu(s_1, e, x) = \nu_2 = \langle e, \sigma_2, \tau_0 \rangle$. But $\tau_0 \subseteq \tau_1$, and hence at stage $s+1$ by Rule R_3, $\nu_3 = \langle e, \sigma_2, \tau \rangle$ is checked, and x is placed on track C_3 with $\nu(s_1+1, e, x) = \nu_3$, contrary to (5.7)

5.7. *Rule $\hat{R}_4$ and the key Lemma* 5.14

The next Rule $\hat{R}_4$ determines enumeration of elements $\hat{y}$ into sets $\hat{U}_n$ when $\hat{y}$ is in pocket $\hat{Q}$. Rule R_4 and the corresponding lemmas are entirely dual and will be omitted.

Rule $\hat{R}_4$. Suppose that element $\hat{x}$ is in pocket $\hat{Q}$ at the end of step 1 of the construction at stage $s+1$. Then $\hat{x}$ remains in $\hat{Q}$ through the end of stage $s+1$. Furthermore, $\tau(s+1, x, \hat{x}) = \tau(s, x, \hat{x})$ else $\hat{x}$ was removed from $\hat{Q}$ under Rule $\hat{R}_8$ during step 1 of the construction.

Case 1. If $\nu^*(s, \hat{x}) \in \mathcal{M}_s$, do nothing.

Case 2. Otherwise perform the following enumeration on $\hat{x}$. Let $\mathcal{S}_{\leqslant t}(D)$ denote the sequence which is the concatenation of the sequences $\{\mathcal{S}_u(D): u \leqslant t\}$. Let $\nu'(s+1) = \langle d, \sigma', \tau' \rangle$ denote $\nu^*(s, \hat{x})$. Define

$$\mathcal{T}_{s+1} = \{\nu: \nu \in \mathcal{M}_s \text{ and } \nu'(s+1) \leqslant^{\tau} \nu\}.$$

Note that $\nu'(s+1) \in \mathcal{P}_s$, and thus $\mathcal{T}_{s+1} \neq \emptyset$, by the definitions of $\mathcal{P}_s$ and d. Furthermore, for each $\nu \in \mathcal{M}_s$, $\nu \in \mathcal{S}_{\leqslant s}(D)$. Define $\nu''(s+1) = \langle d, \sigma'', \tau \rangle$ to be the last ν on the sequence $\mathcal{S}_{\leqslant s}(D)$ such that $\nu \in \mathcal{T}_{s+1}$. Enumerate $\hat{x}$ in $\hat{U}_{n,s+1}$ for each $n \in \sigma'' - \sigma'$.

Hence, in either case,

$$\nu(s+1, d, \hat{x}) = \nu''(s+1) \in \mathcal{M}_s.$$

(This Rule $\hat{R}_4$ is a combination of the old Rules $\hat{R}_4$ and $\hat{R}_5$ of [33]. There is now no Rule $\hat{R}_5$, although we have kept the numbering of rules as in [33] to avoid confusion.) Rule $\hat{R}_4$ immediately yields the following lemmas.

Lemmas 5.11. $(\forall s)(\forall \hat{y})[\hat{y} \in \hat{Q}_{s+1} \Rightarrow \nu(s+1,\ l(s, \hat{y}), \hat{y}) \in \mathcal{M}_s]$.

Proof. Immediate by Rule $\hat{R}_4$.

Lemma 5.12. $(\forall \hat{y})[\hat{y} \in \hat{Q}_\omega \Rightarrow \nu^*(\hat{y}) \in \mathcal{M}_\omega]$.

Proof. Fix $\hat{y} \in \hat{Q}_\omega$. By (5.6) let $d = \lim_s d(s, \hat{y})$. Choose s_1 such that $\nu(s, d, \hat{y}) = \nu^*(\hat{y})$ for all $s \geqslant s_1$. Now by Lemma 5.11, $\nu^*(\hat{y}) \in \mathcal{M}_s$ for all $s \geqslant s_1$.

On the other hand, Rule $\hat{R}_4$ has not allowed too much enumeration for we shall now prove by a series of lemmas that C covers any stream $\hat{X}$ of M. The hypothesis that each $\nu \in \mathcal{K}_\omega$ is excluded from $\mathcal{M}_{s+1}$ finitely often under Condition 3 will be discharged in Section 6.

Lemma 5.13. *Assume that every $\nu \in \mathcal{K}_\omega$ is excluded from $\mathcal{M}_{s+1}$ for at most finitely many s under Condition 3 of Definition 5.5. Fix ν_1 and ν_2. Suppose that D τ-exactly covers ν_1, and that Rule $\hat{R}_4$ applies at infinitely many stages s such that $\nu'(s) = \nu_1$ and $\nu''(s) = \nu_2$. Then D τ-exactly covers ν_2 also.*

Proof. Fix $\nu_1 = \langle e, \sigma_1, \tau_1 \rangle$, and $\nu_2 = \langle e, \sigma_2, \tau_1 \rangle$ satisfying the hypotheses, and assume to the contrary that D fails to τ-exactly cover ν_2. Then there exists s_1 such that

$$(\forall s)_{\geq s_1}(\forall \nu)[\nu_2 \leq^\tau [\nu]_e \Rightarrow \nu \notin \mathcal{S}_s(D)].$$

But $\nu''(s) = \nu_2$ for infinitely many s implies that $\nu_2 \in \mathcal{M}_s$ for infinitely many s. Thus, for some $s_2 \geq s_1$, $\nu_2 \in \mathcal{M}_s$ for all $s \geq s_2$.

On the other hand, some $\nu_3 \in \mathcal{S}(D)$ i.o. where $\nu_1 \leq^\tau \nu_3$ and $\nu_3 \in \mathcal{K}_\omega$ because D τ-exactly covers ν_1. Furthermore, ν_3 cannot be excluded from $\mathcal{M}_s$ at any $s \geq s_2$ by Condition 2 else ν_2 is excluded from $\mathcal{M}_s$ also. But $\nu_3 \in \mathcal{K}_\omega$ implies that ν_3 is excluded from $\mathcal{M}_s$ at most finitely often under Conditions 1 or 3. Thus $\nu_3 \in \mathcal{M}_\omega$ because $\nu_3 \in \mathcal{S}(D)$ i.o. Furthermore, $\nu_2 \notin \mathcal{S}(D)$ i.o. implies that for almost all s, some occurrence of ν_3 follows the last occurrence of ν_2 on the sequence $\mathcal{S}_{\leq s}(D)$. Thus, for almost all s if $\nu'(s) = \nu_1$ under Rule $\hat{R}_4$, we prefer ν_3 to ν_2 in the definition of $\nu''(s)$, contrary to hypothesis.

Along with Lemma 5.11, the following lemma is crucial in Section 6 because it implies that for every e there are only finitely many elements $\hat{y} \in \hat{M}$ such that $|\nu^*(\hat{y})| < e$, and hence almost every element $\hat{y}$ will have to respect the correspondences $U_e \leftrightarrow \hat{U}_e$ and $\hat{V}_e \leftrightarrow V_e$ in choosing a preimage $x \in M$. The main idea in the proof of Lemma 5.14 is that when some $\nu_1 \in \mathcal{S}_s(\hat{X}) - \mathcal{P}_s$, we do two things. First we exclude from $\mathcal{M}_{s+1}$ under Condition 2 all ν of length $> e_1 = |\nu_1|$, which causes $d(s+1, \hat{y}) \leq e_1$. Secondly, we add to $\mathcal{K}_s$ all ν_2 such that $\nu_1 \leq^\tau \nu_2$ which tends to force D to τ-exactly cover ν_1 if C covers ν_1. These two features enable us to apply Lemma 5.13.

Lemma 5.14. *Assume that every $\nu \in \mathcal{K}_\omega$ is excluded from $\mathcal{M}_{s+1}$ for at most finitely many s under Condition 3 of Definition 5.5(a). For each track X of M ($\hat{X}$ of $\hat{M}$) and each ν,*

(i) $\nu \in \mathcal{S}(\hat{X})$ *i.o.* $\Rightarrow \nu \in \mathcal{P}_\omega$, *and*

(ii) $\nu \in \mathcal{S}(X)$ *i.o.* $\Rightarrow \nu \in \mathcal{P}'_\omega$.

Proof. The proof is by induction on the length of ν. Fix e and assume (i) and (ii) for all ν such that $|\nu| < e$. It suffices to prove (i) for all ν of length e since (ii) is dual. By inductive hypotheses (ii) for ν of length $<e$, each ν of length $<e$ is added to $\mathcal{H}'$ only finitely often. Thus, by Lemma 5.9,

$$C \text{ covers } \mathcal{S}^e(\hat{C}_3). \tag{5.8}$$

Now by inductive hypothesis (i), each ν of length e is excluded from $\mathcal{M}_{s+1}$ under Condition 2 of Definition 5.5(a) for at most finitely many s. Thus, by the proof of Lemma 5.7(i) we have for every ν of length e,

$$\nu \in \mathcal{S}(D) \text{ i.o.} \quad \text{and} \quad \nu \in \mathcal{K}_\omega \Rightarrow \nu \in \mathcal{M}_\omega \tag{5.9}$$

and

$$\nu \in \mathcal{P}_s \text{ for infinitely many } s \Rightarrow \nu \in \mathcal{P}_\omega. \tag{5.10}$$

Now assume for a contradiction that for some $\hat{X}$ and ν, $\nu_1 \in \mathcal{S}(\hat{X})$ i.o., but $\nu_1 \notin \mathcal{P}$. Let $\nu_1 = \langle e, \sigma_1, \tau_1 \rangle$ with σ_1 minimal for e, and τ_1 minimal for e and σ_1. By (5.10), $\nu_1 \in \mathcal{P}_s$ for finitely many s, and thus

$$\nu_1 \in \mathcal{S}_s(\hat{X}) - \mathcal{P}_s \quad \text{for infinitely many } s. \tag{5.11}$$

But for each such s of (5.11), all ν' of length $>e$ are excluded from $\mathcal{M}_{s+}$ under Condition 2 and hence

$$\nu \in \mathcal{M}_\omega \Rightarrow |\nu| \leq e. \tag{5.12}$$

Now choose infinitely many elements v_j, $j \in N$, and corresponding states $s_j + 1$ such that for all $j \in N$,

$$\nu_1 = \nu(s_j + 1, \hat{y}_j) \neq \nu(s_j, e, \hat{y}_j). \tag{5.13}$$

Note that s_j exists because $\nu(v_j, e, \hat{y}_j) = \langle e, \emptyset, \emptyset \rangle \in \mathcal{P}_\omega$, where $\hat{y}_j$ enters $\hat{M}$ at stage v_j. For each $j \in N$, define the finite sequence of full e-states,

$$\mathcal{T}_j = \{\nu : (\exists s)[v_j \leq s \leq s_j + 1 \text{ and } \nu = \nu(s, e, \hat{y}_j)]\}.$$

Let $\mathcal{T}$ be the concatenation of $\{\mathcal{T}_j : j \in N\}$. Clearly, $\nu_1 \in \mathcal{T}$ i.o., Now if some $\nu \in \mathcal{T}$ i.o., then from (5.11) and the definition of $\mathcal{H}$, note that $\nu' \in \mathcal{H}$ i.o. for all ν' such that $\nu \leq^\tau \nu'$. From this and Rule R_3 we have

$$\nu \in \mathcal{T} \text{ i.o.} \quad \text{and} \tag{5.14}$$

But $\nu_1 \notin \mathcal{P}_\omega$, and thus by (5.10), $\nu_1 \notin \mathcal{P}_s$ for infinitely many s, so D fails to τ-exactly cover ν_1. Thus, by (5.14) we have,

$$C \text{ does not cov} \tag{5.15}$$

On the other hand, we get a contradiction from (5.15) by proving that

$$C \text{ covers } \mathcal{T}, \tag{5.16}$$

and a fortiori that

$$C \text{ covers } \nu_1. \tag{5.17}$$

First note that by the minimality of σ_1 and τ_1 above, we have

$$\nu \in \mathcal{T} \text{ i.o. } \quad \text{and} \quad \nu \neq \nu_1 \Rightarrow \nu \in \mathcal{P}_\omega, \tag{5.18}$$

and thus

$$(\text{a.e. } j)(\forall s)_{\leq s_j}[d(s, \hat{y}_j) \geq e], \tag{5.19}$$

by the definition of $d(s, \hat{y})$ from $\mathcal{P}_s$, by (5.18) and by induction on t for $v_j \leq t \leq s_j$, where $v_j = (\mu t)[\hat{y}_j \in \hat{M}_t]$.

Assume for a contradiction that (5.16) fails, and choose $\nu_0 = \langle e, \sigma_0, \tau_0 \rangle$ such that

$$\nu_0 \in \mathcal{T} \text{ i.o. } \quad \text{but} \quad C \text{ does not cover } \nu_0, \tag{5.20}$$

where σ_0 and τ_0 are minimal. Choose an infinite set J and stages $t_j + 1 \leq s_j + 1$ such that

$$(\forall j \in J)[\nu_0 = \nu(t_j + 1, e, \hat{y}_j) \neq \nu(t_j, e, \hat{y}_j)]. \tag{5.21}$$

(Of course, $t_j + 1 > v_j$ since C covers $\hat{C}_0$.) By the minimality of σ_0, τ_0 and by (5.21),

$$C \text{ covers the sequence } \{\nu(t_j, e, \hat{y}_j) : j \in J\}. \tag{5.22}$$

Now by (5.20), (5.21), (5.22), and Rule $\hat{\mathrm{R}}_6$, for almost all $j \in J$, either Rule $\hat{\mathrm{R}}_3$ or $\hat{\mathrm{R}}_4$ applies to $\hat{y}_j$ at stage $t_j + 1$. By (5.20) and (5.8), $\hat{\mathrm{R}}_3$ applies for at most finitely many $j \in J$.

Thus, for almost all $j \in J$, Rule $\hat{\mathrm{R}}_4$ applies to $\hat{y}_j$ at stage $t_j + 1$, with $\nu_0 < \nu''(t_j + 1)$ by (5.19). However, $\mathcal{M}_s[\nu_0] \subseteq \{\nu_0\}$ for almost all s by (5.12) and the fact that D cannot τ-exactly cover ν_0. Hence, $\nu''(t_j + 1) = \nu_0$ for almost all $j \in J$. Fix any ν_0' such that $\nu_0' = \nu'(t_j + 1)$ in Rule $\hat{\mathrm{R}}_4$ for infinitely many $j \in J$. By (5.22), C covers ν_0', and thus by (5.14), D τ-exactly covers ν_0'. Thus, by Lemma 5.13, D τ-exactly covers ν_0, and therefore C covers ν_0 contrary to (5.20).

The main technical device not found in [33] for dealing with the case where elements are removed from M is embodied in the extra exclusion from $\mathcal{M}$ under Condition 3 of Definition 5.5 and the assignment and motion of the markers which prevent too much $\mathcal{M}$ exclusion. This device was first used to prove the present theorem. Then Stob used the device in a different way in his automorphism theorem [39, 40]. His Condition 3 exclusion from $\mathcal{M}$ is essentially the same, but the assignment and motion of his markers is quite different. This paper and Stob's were written approximately simultaneously stressing those parts which were similar to each other and to [33] so as to develop a common framework for these automorphism arguments, analogous to the framework for the infinite injury priority method [34].

6. The verification

6.1. *The Finiteness Lemma*

The next lemma is crucial for constructing the permutation from $\bar{A}$ to N. The statement is the same as in [24, Lemma 6.1], but the proof is more complicated than before because an element $x = p(s, \nu_0)$ may now enter A thereby leaving M. If $\hat{q}(\nu_0) = \lim_s \hat{q}(s, \nu_0)$ exists (i.e., ν_0 is *right stable*), but $p(\nu_0)$ does not (ν_0 is *left unstable*), we wish to arrange that $\hat{q}(\nu)$ exists for only finitely many $\nu \geqslant \nu_0$, (i.e., $\hat{Q}[\nu_0]$ is finite). This is done by arranging that $\mathcal{M}_\omega[\nu_0]$ is finite since if $\hat{y} \in \hat{Q}_s$, $\nu^*(s, \hat{y}) \in \mathcal{M}_s$. As in [33], Condition 1 exclusion handles the case where $p(s, \nu_0)$ becomes undefined due to changes in $|\nu_0|$-state. Condition 3 and the definition of x' in case 2 of the construction handle the case in which $p(s, \nu_0)$ becomes undefined because $p(s, \nu_0)$ enters A.

Lemma 6.1 (Finiteness Lemma). *Fix ν_0 and suppose that*

(i) $(\forall \nu < \nu_0)[\lim_s p(s, \nu)$ *exists*],

(ii) $(\forall \nu \leqslant \nu_0)[\lim_s \hat{q}(s, \nu)$ *exists*], *and*

(iii) $\lim_s p(s, \nu_0)$ *does not exist.*

Then $\mathcal{M}_\omega[\nu_0]$ is finite.

Proof. Suppose that $\nu_0 = \langle e_0, \sigma_0, \tau_0 \rangle$ and let s_0 be a stage such that

$$(\forall \nu < \nu_0)[p(s_0, \nu) = p(\nu)] \tag{6.1}$$

and

$$(\forall \nu \leqslant \nu_0)[\hat{q}(s_0, \nu) = \hat{q}(\nu)]. \tag{6.2}$$

We show that, if $\nu_1 \in \mathcal{M}_{s_1}[\nu_0] - \mathcal{M}_{s_1-1}[\nu_0]$ for some $s_1 > s_0$ and $r(s_1, \nu_1) \geqslant s_0$, then ν_1 is excluded from $\mathcal{M}_{t+1}$ for some stage $t \geqslant s_1$. (This clearly implies that $\mathcal{M}_\omega[\nu_0]$ is finite since $\mathcal{M}_\omega[\nu_0] \subseteq \mathcal{M}_{s_0}[\nu_0] \cup \{\nu \colon r(s_0, \nu) < s_0\}$.) If ν_1 is never excluded from $\mathcal{M}_{t+1}$ for $t \geqslant s_1$ by Condition 1 of Definition 5.5, then it must be the case that

$$\lim_s r(s, \nu) = r(s_1, \nu) \tag{6.3}$$

for all ν such that $r(s_1, \nu) \leqslant r(s_1, \nu_1)$.

Since $\nu_1 \in \mathcal{M}_{s_1} - \mathcal{M}_{s_1-1}$, some element x is on track D at stage s_1 in state $\nu(s_1, x, x)$ such that

$$\nu_0 \leqslant \nu_1 \leqslant \nu(s_1, x, x).$$

By Rule R_2, x is preferred as $p(s_1 + 1, \nu_0)$ unless $\rho(s_1, \nu_0, p(s_1, \nu_0)) \leqslant r(s_1, \nu_1)$. Let $k_1 = r(s_1, \nu_1)$. Thus, in any case,

$$\rho(s_1 + 1, \nu_0, p(s_1 + 1, \nu_0)) \leqslant r(s_1 + 1, \nu_1) = k_1. \tag{6.4}$$

Now by Rule R_{12}, $p(s, \nu_0) = p(s + 1, \nu_0)$ unless Rule R_2, R_{10}, R_{11}, or R_A removes $p(s, \nu_0)$ from P. Now at a stage $s > s_0$, Rule R_{10}(c) cannot apply to $p(s, \nu_0)$ by (6.1), and Rule R_{11} cannot apply to $p(s, \nu_0)$ by (6.2). (Rule R_{10}(b), of course, never removes $p(s, \nu_0)$.) Suppose Rule R_2 is used to replace $p(s, \nu_0)$ at some stage $s_2 + 1 > s_1 + 1$ before any other rules remove $p(s, \nu_0)$. Then it follows by induction on stages s, $s_1 + 1 \leqslant s \leqslant s_2$, that

$$\rho(s_2, \nu_0, p(s_2, \nu_0)) = \rho(s_1 + 1, \nu_0, p(s_1 + 1, \nu_0)) \leqslant k_1. \tag{6.5}$$

This follows by (6.3), (6.4) and the fact that $p(s_2, \nu_0) = p(s_1 + 1, \nu_0)$ does not change in e-state for $e=|\nu_0|$ while in pocket P. By (6.1), $p(s_2, \nu_0)$ is removed only because Rule R_2 prefers $p(s_2+1, \nu_0)$ and hence,

$$\rho(s_2, \nu_0, p(s_2+1, \nu_0)) < \rho(s_2, \nu_0, p(s_2, \nu_0)) \leq k_1. \tag{6.6}$$

Thus, by induction on stages $s > s_1$, $\rho(s, \nu_0, p(s, \nu_0))$ decreases with each application of Rule R_2 on $p(s, \nu_0)$ so long as no other rule applies to $p(s, \nu_0)$.

But then since $\lim_s p(s, \nu_0)$ fails to exist, there must be a stage $s+1 > s_1+1$ at which either R_A or R_{10}(a) first removes $p(s, \nu_0)$ from P. Let $x = p(s, \nu_0)$, $k_2 = \rho(s, \nu_0, x)$, and ν_2 be such that $r(s, \nu_2) = k_2$. Note that $k_2 \leq k_1$. If Rule R_A applies to x at stage $s+1$, then $x \in U_{1,s+1}$ so any ν such that $r(s, \nu) \geq \max\{k_2, s_0\}$ is excluded from $\mathcal{M}_{s+1}$ under Condition 3 of Definition 5.4. In particular, ν_1 is excluded since $r(s, \nu_1) = r(s_1, \nu_1) = k_1 \geq \max\{k_2, s_0\}$.

If Rule R_{10}(a) applies to x at stage $s+1$ then x is placed above hole H_2 by Rule R_{12} since $\nu_2 <^\tau \nu_3 = \nu(s+1, e, x)$ where $e = |\nu_2|$. Now by the definition of x' in case 2 of the construction, $x' = x$, and x is immediately processed at stage $s+2$, and either: (1) x is enumerated in $U_{1,s+2}$ in which case ν_1 is excluded under Condition 3 just as if R_A had applied to x at stage $s+1$; or (2) x enters track C at stage $s+2$ in which case ν_2 is reset on $\mathcal{K}$ at stage $s+2$, because $\nu_2 <^\tau \nu_3 = \nu(s+2, e, x)$, so $r(s+2, \nu_2) > r(s+1, \nu_2)$ contrary to (6.3).

6.2. *Finitely much exclusion from $\mathcal{M}$ under Condition* 3

In order to prove Lemma 6.1 it was necessary to alter the construction by adding extra exclusion from $\mathcal{M}_{s+1}$ under Condition 3. We now prove that every $\nu \in \mathcal{K}_\omega$ is excluded under Condition 3 at most finitely often. This will discharge the extra hypothesis of Lemmas 5.7, 5.13 and the crucial Lemma 5.14. Condition 3 exclusion works roughly as follows. When an element x enters pocket P and becomes $p(s, \nu_0)$, it is assigned a marker $\Gamma_{\langle k,l,\nu_0\rangle}$ under Rule R_2. Condition 3 exclusion applies only to those ν such that $r(t, \nu) \geq \max\{k, l\}$, and x leaves P at stage $t+1$ and enters A (i.e., $x \in U_{1,t+2}$). However, our oracle procedure of case 2 of the construction guarantees that this can happen only finitely often for a fixed k, l, and ν_0, because x was enumerated in the set $Z_{\langle k,l,\nu_0\rangle}$ at some stage $u < t$ corresponding to marker $\Gamma_{\langle k,l,\nu_0\rangle}$ before x was released from hole H_2 to enter P. The latter occurred only if the oracle function h for $\bar{A}$ 'permitted' in the sense that $h(\theta(j, u), t) = 1$ where $j = \langle k, l, \nu\rangle$, and t is as in (4.2).

Lemma 6.2. *Every $\nu_0 \in \mathcal{K}_\omega$ is excluded from $\mathcal{M}_{s+1}$ for at most finitely many s under Condition* 3 *of Definition* 5.5.

Proof. Let ν^k denote the unique ν such that $r(s, \nu) = k$ for some s. Fix $\nu_0 \in \mathcal{K}_\omega$. Choose s_0 such that if $j = r(s_0, \nu_0)$, then

$$(\forall \nu)[r(s_0, \nu) \leq j \Rightarrow r(s_0, \nu) = \lim_s r(s, \nu)]. \tag{6.7}$$

If Rule R_{10}(a) applies to some $x = p(s, \nu)$ at some stage $s+1 > s_0$, then by case 2 of the construction either $x \in U_{1,s+2}$, in which case for notational convenience we say *Rule* $R^{+}_{10}(a)$ *applies to* x, or else x is placed on track C at stage $s+2$, in which case we say *Rule* $R^{-}_{10}(a)$ *applies to* x. (Think of $R^{+}_{10}(a)$ as roughly equivalent to R_A and $R^{-}_{10}(a)$ as the usual $R_{10}(a)$.) In the latter case, if $k=\rho(s,\nu,p(s,\nu))$, then $\nu^k \in \mathcal{K}^{*2}_{s+2}$ and hence $r(s+2,\nu^k)>r(s,\nu^k)$. Hence, by (6.7), Rule $R^{-}_{10}(a)$ cannot apply at stage $s+1>s_0$ to any $x=p(s,\nu)$ such that $\rho(s,\nu,p(s,\nu))\leqslant j$.

Now suppose for a reductio ad absurdum that ν_0 is excluded infinitely often from $\mathcal{M}_{s+1}$ under Condition 3. Then by Definition 5.5 there is some ν_1 of minimal length such that

$$j \geqslant l = (\mu s)(\forall\, \nu \leqslant \nu_1)[\hat{q}(\nu,s)=\hat{q}(\nu)] \tag{6.8}$$

and

$$(\exists\, k \leqslant j)(\exists^\infty s)[\text{either Rule } R^{+}_{10}(a) \text{ or } R_A \text{ removes } p(s,\nu_1) \text{ from } P \text{ at stage } s+1, \text{ and } \rho(s,\nu_1,p(s,\nu_1))=k]. \tag{6.9}$$

Choose $s_1>\max\{s_0,l\}$ such that if $x=p(s,\nu_1)$ for $s\geqslant s_1$, then x last entered P at some stage $>s_0$. Consider any element $x=p(s,\nu_1)$, $s>s_1$, to which (6.9) applies, and which last entered P at some stage $u>s_0$. Then some marker $\Gamma_{\langle k',l,\nu_1\rangle}$ was assigned to x at stage u, where $k'=\rho(u,\nu,x)$. Now by (6.7), Rule $R_{10}(b)$, and the remarks at the end of Section 5.4, $\rho(t,\nu_1,p(s,\nu_1))=k'$ for all t, $u\leqslant t\leqslant s$, so $k'=k$ by (6.5). Hence, by assuming $s\geqslant s_1$ from now on, we may assume in (6.9) that $p(s,\nu_1)$ is associated with marker $\Gamma_{\langle k,l,\nu_1\rangle}$ at stage s.

Now by Rule R_{12}, any element $p(s,\nu)\in P_s$ can be removed from P only by an application of Rule R_A, R_2, R_{10}, or R_{11}. By the definition of l, $p(s,\nu_1)$ cannot be removed from P by Rule R_{11} at any stage $s+1>s_1$. (Recall that Rule $R_{10}(b)$ never removes any element.) We now wish to show that $p(s,\nu_1)$ can be removed only by Rule R_A, R_2, or $R^{+}_{10}(a)$, i.e., we must show that $p(s,\nu_1)$ cannot be removed by Rule $R_{10}(c)$ because of action of some $\nu_2<\nu_1$.

Claim. $(\forall\, \nu<\nu_1)[\lim_s p(s,\nu)$ *exists*$]$.

Proof. If not, choose $\nu_2<\nu_1$ of minimal length such that $\lim_s p(s,\nu_2)$ diverges. Choose $s_2\geqslant s_1$ such that $p(s_2,\nu)=\lim_s p(s,\nu)$ for all $\nu<\nu_2$. Now

$$(\exists^\infty s>s_2)[p(s,\nu_2) \text{ is removed from } P \text{ at stage } s+1]. \tag{6.10}$$

By choice of l, this removal cannot be under Rule R_{11}, and by choice of s_2 it cannot be under Rule $R_{10}(c)$. Now by the same argument as in the proof of Lemma 6.1, there can be only finitely many removals under Rule R_2 until a removal under either Rule $R_{10}(a)$ or Rule R_A. Hence,

$$(\exists k_1)(\exists^\infty s>s_2)[\text{either Rule } R_{10}(a) \text{ or } R_A \text{ removes } p(s,\nu_2) \text{ from } P \text{ at stage } s+1, \text{ and } \rho(s,\nu_2,p(s,\nu_2)=k_1]. \tag{6.11}$$

Choose k_1 minimal satisfying (6.11). Note that $k_1\leqslant k\leqslant j$ by (5.5), (6.9), and (6.7) because any x entering P at stage $t+1$ to become $p(t+1,\nu_1)$ was first eligible to become $p(t+1, \nu_2)$ under Rule R_2 because $\nu_2 < \nu_1$, and therefore $\rho(u, \nu_2, p(u, \nu_2)) \leq \rho(u, \nu_1, p(u, \nu_1))$ at all stages $u \geq t$ during which $p(u, \nu_1)$ and $p(u, \nu_2)$ remain in P. But since $k_1 \leq j$, we know that $R_{10}^-(a)$ cannot apply in (6.11) else ν^{k_1} is reset on $\mathcal{K}$ contrary to (6.7). Hence, $R_{10}(a)$ in (6.11) may be replaced by $R_{10}^+(a)$. Now define

$$l_1 = (\mu s)(\forall \nu < \nu_2)[\hat{q}(s, \nu) = \hat{q}(\nu)].$$

Note that $l_1 < l$ by the Definition 5.2 of $\hat{q}(s, \nu)$ because $\hat{q}(s, \nu_2)$ is defined whenever $\hat{q}(s, \nu_1)$ is defined. But since $k_1, l_1 < j$, (6.11) contradicts the minimality of $|\nu_1|$ for ν_1 satisfying (6.8) and (6.9). This completes the proof of the claim.

It therefore follows from the claim that $p(s, \nu_1)$ can be removed from P at most finitely often under Rule $R_{10}(c)$. Now using this and (6.7), choose $s_2 > s_1$ such that if $\rho(s, \nu_1, p(s, \nu_1)) \leq j$ then $p(s, \nu_1)$ is not removed from P at stage $s+1 > s_2$ by Rule $R_{10}^-(a)$, $R_{10}(c)$, or R_{11}, i.e., only Rule R_2, $R_{10}^+(a)$ or R_A can remove $p(s, \nu_1)$ after stage s_2.

But there exists some $k \leq j$ such that ν^j is excluded from $\mathcal{M}_{s+1}$ under Condition 3 for infinitely many $s > s_2$ such that

$$\rho(s, \nu_1, p(s, \nu_1)) = k \leq j \tag{6.12}$$

and

$$p(s, \nu_1) \in P_s - P_{s+1}. \tag{6.13}$$

Choose k minimal satisfying (6.12) and (6.13) for infinitely many $s \geq s_2$. Choose $s_3 > s_2$ such that at no stage $s+1 \geq s_3$ do (6.12) and (6.13) apply for any $k' < k$. Now by the minimality of k, Rule R_2 cannot apply to $p(s, \nu_1)$ at any stage $s+1 \geq s_3$, satisfying (6.12) and (6.13). Thus, there are infinitely many stages $s \geq s_3$ satisfying (6.12) and (6.13) such that Rule $R_{10}^+(a)$ or R_A applies to $p(s, \nu_1)$ at stage $s+1$. At such a stage $p(s, \nu_1)$ will be associated with the marker $\Gamma_{\langle k,l,\nu_1\rangle}$ because $\rho(t, \nu_1, p(s, \nu_1)) = k$ for all stages t, $u \leq t \leq s$, where x last entered P at stage $u < s$ and was assigned at stage u the marker $\Gamma_{\langle k,l,\nu_1\rangle}$. Now $x \in U_{1,s+2}$ (so $x \in A$) because either Rule R_A or $R_{10}^+(a)$ applies to x.

However, marker $\Gamma_{\langle k,l,\nu_1\rangle}$ is not injured after stage s_3. This is because any element x associated with $\Gamma_{\langle k,l,\nu_1\rangle}$ at some stage $t+1 > s_3$ becomes $x = p(t+1, \nu_1)$, and satisfies (6.12) for $s = t+1$. By the choice of s_3, Rule R_2 cannot remove x from P, nor can Rule $R_{10}^-(a)$. Thus, the association of $\Gamma_{\langle k,l,\nu_1\rangle}$ with x can only be cancelled when x is removed from P at stage $s+1$ under Rule R_A or $R_{10}^+(a)$ in which case $x \in U_{1,s+2}$, so this cancellation does not constitute an injury to $\Gamma_{\langle k,l,\nu_1\rangle}$ according to Definition 5.4.

Let v be the last stage when marker Γ_j was unjured, where $j = \langle k, l, \nu_1\rangle$. Let $W_{\theta(j,s)}$ for $s > v$ be the corresponding set of marker positions as defined in Section 4.3. Now $W_{\theta(j,s)} \cap \bar{A} = \emptyset$ because each element x associated with Γ_j eventually enters A. Hence, $\lim_s h(\theta(j, s), s) = 0$ for $j = \langle k, l, \nu_1\rangle$. But this contradicts the fact that under case 2(a) of the construction, $h(\theta(j, t), t) = 1$ for infinitely many t;

namely, for some $t \geq s+1$ at each stage $s+1$ such that x above hole H_2 is released to the surface of the machine at stage $s+1$, x later enters pocket P, and x is associated with marker Γ_i.

6.3. The covering lemmas

Using Lemma 6.2 to discharge the hypothesis of Lemma 5.14, we can immediately prove that the rules in Section 5 achieved the covering they were designed for.

Lemma 6.3. *C covers $\hat{C}_3$ and $\hat{C}$ dual covers C_3.*

Proof. By Lemmas 5.8, 5.9, 5.14, 6.2 and the definitions of $\mathcal{H}$ and $\mathcal{H}'$.

Lemma 6.4. $(\forall e)(\text{a.e. } s)(\text{a.e. } \hat{y})[\hat{y} \in \hat{M}_s \Rightarrow d(s, \hat{y}) \geq e]$.

Proof. By Lemmas 5.14, 6.2, and Definition 5.5(d) of $d(s, \hat{y})$.

Lemma 6.5. *Given ν_1 and infinitely many elements $\hat{y}_j$, $j \in N$, such that for all $j \in N$ Rule $\hat{R}_4$ applies to $\hat{y}_j$ at say stage s_j+1 with $\nu_1 < \nu(s_j+1, y_j, \hat{y}_j)$, then C covers ν_1.*

Proof. As in [33, Lemma 5.14].

Lemma 6.6. *C covers every stream $\hat{X}$ of $\hat{M}$.*

Proof. By Lemma 6.3 and Lemma 6.5.

Lemma 6.7. *D τ-exactly covers every stream $\hat{X}$ of $\hat{M}$.*

Proof. By Lemma 6.6 and Lemma 5.10.

Only Lemma 6.4 is absolutely necessary for the rest of our proof but Lemmas 6.6 and 6.7 give considerable insight and are necessary for applications. For example, the main result of Maass [14, Theorem 17], namely that two promptly simple semi-low sets are automorphic, requires Lemma 6.6.

It follows from Lemmas 6.7 and 5.10 that D exactly covers $\hat{Q}$ (and hence $\hat{Q}_\omega$) although in Section 6.4 we need a stronger property which will be guaranteed by Lemma 5.12. The dual to Lemma 6.1 is proved exactly as in [33] since there is no Condition 3 exclusion. There is no dual to Lemma 6.2. The obvious duals to Lemmas 6.3–6.7 have similar proofs.

6.4. The permutation from $\bar{A}$ to N

Lemma 6.8. *Fix ν and suppose that there are infinitely many $\hat{y}_j \in \hat{Q}_\omega$, $j \in N$, such that $\nu \leq \nu(y_j, \hat{y}_j)$, $j \in N$. Then $\lim_s p(s, \nu)$ exists.*

Proof. By the definition of ν^* and Lemma 6.4 it is clear that $\nu \leqslant \nu^*(\hat{y}_j)$ for almost every j. Now by Lemma 6.1, if $\lim_s p(s, \nu)$ does not exist, then $\mathcal{M}_\omega[\nu]$ is finite. But if $\hat{y}_j \in \hat{Q}_\omega$, then $\nu^*(\hat{y}_j) \in \mathcal{M}_\omega$ by Lemma 5.12. For any particular ν_0, $\nu^*(\hat{y}_j) = \nu_0$ for only finitely many j. This is a contradiction.

Notice that $U_n =^* W_n$ for all $n > 1$, because any element $x \in W_n$ is eventually enumerated in U_n under case 4(c) of the construction unless $x < n$ or $x = \lim_s p(s, \nu)$ for some ν such that $n > |\nu|$ (see Rule R_{10}(b)). (Note that x cannot be $p(s, \nu)$ infinitely often without being $\lim_s p(s, \nu)$ because if $x = p(s, \nu) \neq p(s+1, \nu)$, then x leaves pocket P. But x re-enters M at most finitely often by Lemma 5.3.) Notice that we are using the fact that if $x \in W_n$, $n > 1$, then g enumerates x in W_n at infinitely many different stages.

In constructing the desired permutation from $\bar{A}$ to N, our intention is to match $\hat{Q}_\omega$ with P_ω by sending $\hat{q}(\nu)$ to $p(\nu)$ (and dually to match Q_ω to $\hat{P}_\omega$ by sending $q(\nu)$ to $\hat{p}(\nu)$). (Note that $\bar{A}$ is the disjoint union $P_\omega \cup Q_\omega$ and $N = \hat{P}_\omega \cup \hat{Q}_\omega$.) Unfortunately, this fails if $p(\nu)$ does not exist, but then Lemma 6.1 suggests a finite–one map.

Lemma 6.9. *There are finite-one maps $\psi_1 : \hat{Q}_\omega \to P_\omega$ and $\psi_2 : Q_\omega \to \hat{P}_\omega$ such that for every n*

(a) $\psi_1(\hat{Q}_\omega \cap \hat{U}_n) =^* P_\omega \cap U_n$,
(b) $\psi_1(\hat{Q}_\omega \cap V_n) =^* P_\omega \cap \hat{V}_n$,
(c) $\psi_2(Q_\omega \cap \hat{V}_n) =^* \hat{P}_\omega \cap V_n$, and
(d) $\psi_2(Q_\omega \cap U_n) =^* \hat{P}_\omega \cap \hat{U}_n$.

Proof. We define ψ_1 since ψ_2 is similar. If $p(\nu)$ is defined, set $\psi_1(\hat{q}(\nu)) = p(\nu)$. If $p(\nu)$ is undefined, let $\nu' \leqslant \nu$ be the maximal full e-state in the ordering $\leqslant$ such that $p(\nu')$ is defined and set $\psi_1(\hat{q}(\nu)) = p(\nu')$. This map is finite–one by Lemma 6.1 and Lemma 6.4, and satisfies (a) and (b) since $p(\nu)$ and $\hat{q}(\nu)$ are in the same full $|\nu|$-state.

The map ψ_1 is sufficient to induce an automorphism of $\mathcal{L}^*(A)$ to $\mathcal{E}^*$. One can also use ψ_1 to construct a 1–1 map ψ from $\bar{A}$ to N using [33, Corollary 1.7, p. 85].

7. Conclusion and open questions

We have shown that if A is r.e. and $\bar{A}$ is infinite and semi-low, then $\mathcal{E}_A^* \cong^{\text{eff}} \mathcal{E}^*$, where $\cong^{\text{eff}}$ denotes that the isomorphism $\Phi : \mathcal{E}_A^* \cong \mathcal{E}^*$ is effective in the sense of Section 3.1, i.e., $\Phi((W_n \cap \bar{A}) = W^*_{f(n)}$ for some recursive permutation f of N. With a few changes in the construction and proofs, the same conclusion can be obtained without the hypothesis that A is r.e. (Note that if $\bar{A}$ is semi-low, then $\bar{A} \leqslant_T \emptyset'$ so $\bar{A} = \lim_s B_s$ for $\{B_s\}_{s\in N}$ a recursive sequence of recursive sets.)

Lachlan [8] generalized Robinson's result (Theorem 2.1) by proving that if A is r.e. coinfinite and low$_2$ (namely, $A'' \equiv_T \emptyset''$) then A has a maximal superset. Bennison and Soare derived the same conclusion under the hypothesis '$\bar{A}$ is semi-low$_{1.5}$', namely

$$\{e: W_e \cap \bar{A} \text{ is infinite}\} \leqslant_m \{e: W_e \text{ is infinite}\}.$$

(These two hypotheses are mutaully incomparable but each includes the case 'A is low$_1$'.) It is an open question whether either of these hypotheses suffices in Theorem 1.1 in place of '$\bar{A}$ semi-low'. In the case of low$_2$, we could not hope to achieve $\cong^{\text{eff}}$, so any proof would presumably begin by working only on some appropriate skeleton of the r.e. sets as in [33]. A first step in this direction would be to prove that for any coinfinite low r.e. set A and any r.e. set B there exists $C \supseteq A$ such that $\mathscr{L}^*(C) \cong \mathscr{L}^*(B)$ but this is unknown even if $\mathscr{L}^*(C)$ is a Boolean algebra.

Also interesting is the open question of characterizing those coinfinite r.e. sets A such that $\mathscr{E}^*_{\bar{A}} \cong^{\text{eff}} \mathscr{E}^*$. We know only that

$$\bar{A} \text{ semi-low} \Rightarrow \mathscr{E}^*_{\bar{A}} \cong^{\text{eff}} \mathscr{E}^* \Rightarrow \bar{A} \text{ semi-low}_{1.5}.$$

(The second implication is trivial.) It seems unlikely that either implication can be reversed but this is unknown.

These problems are also related to the important question of invariant degree classes. A class $\boldsymbol{C}$ of non-zero r.e. degrees is *invariant* if $\boldsymbol{C} = \{\deg(W): W \in \mathscr{C}\}$ for some class $\mathscr{C}$ $\mathscr{E}$ invariant under Aut $\mathscr{E}$. For every $n \geqslant 0$ define the subclasses of the r.e. degrees $\boldsymbol{R}$,

$$\boldsymbol{H}_n = \{\boldsymbol{d}: \boldsymbol{d} \in \boldsymbol{R} \text{ and } \boldsymbol{d}^{(n)} = \boldsymbol{O}^{(n+1)}\}$$

$$\boldsymbol{L}_n = \{\boldsymbol{d}: \boldsymbol{d} \in \boldsymbol{R} \text{ and } \boldsymbol{d}^{(n)} = \boldsymbol{O}^{(n)}\}$$

where $\boldsymbol{d}^{(0)} = \boldsymbol{d}$, and $\bar{\boldsymbol{L}}_n = \boldsymbol{R} - \boldsymbol{L}_n$. Martin's theorem that the degrees of maximal sets constitute $\boldsymbol{H}_1$ shows that $\boldsymbol{H}_1$ is invariant. Lachlan [8] and Shoenfield [30] showed that $\bar{\boldsymbol{L}}_2$ is invariant, being the class of coinfinite r.e. sets with no maximal superset. Thus, the known invariant classes are $\bar{\boldsymbol{L}}_0$, $\boldsymbol{H}_1$, and $\bar{\boldsymbol{L}}_2$. Martin conjectured that among the classes $\boldsymbol{H}_n, \bar{\boldsymbol{L}}_n$ for $n \geqslant 0$, the invariant classes are precisely H_{2n+1}, and $\bar{\boldsymbol{L}}_{2n}$ for all $n \geqslant 0$. In particular, it is unknown whether $\bar{\boldsymbol{L}}_1$ is invariant. If so, then we must find some property invariant under Aut $\mathscr{E}$ shared by every coinfinite set A in any degree $\boldsymbol{a} \in \boldsymbol{L}_1$ but not by every set A in any degree $\boldsymbol{a} \in \bar{\boldsymbol{L}}_1$. The property $\mathscr{E}^*(\bar{A}) \cong \mathscr{E}$ might be such a property. If $\boldsymbol{L}_1$ is not invariant, then perhaps the present automorphism machinery could be used to show that every low$_2$ r.e. set A could be carried to some low$_1$ r.e. set B by some $\Phi \in$ Aut $\mathscr{E}$. If true this would imply $\mathscr{E}^*_{\bar{A}} \cong \mathscr{E}^*$ for every coinfinite low$_2$ set.

The ultimate goal is to find complete sets of invariants for classifying the orbits of r.e. sets under Aut $\mathscr{E}$. A coinfinite set is *simple* if $\bar{A}$ contains no infinite r.e. set. Let A and B be low simple sets. Since their principal filters are isomorphic by Theorem 1.1, it seemed possible that A and B might be automorphic. However, there is a stronger property than simplicity known as *d-simplicity* which is invariant under Aut $\mathscr{E}$ and which is possessed by some but not all low simple sets [13]. A major open question is to classify the orbits of low simple sets, and indeed to find complete sets of invariants for one such orbit. In [15] it is shown that d-simplicity (and lowness) will not suffice. A splitting property is introduced there and it is left open whether this property suffices. Of course, by Maass's theorem [14], any two promptly simple low sets are automorphic but prompt simplicity is not invariant under automorphisms.

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Added in proof

W. Maass has recently answered one of the above questions by proving that if A is r.e. and $\bar{A}$ is infinite and semi-low$_{1.5}$ then $\mathscr{E}_A^* \cong^{\text{eff}} \mathscr{E}^*$, thereby giving a very pleasing characterization of the latter property. Maass's proof uses the automorphism method for low sets here and the methods of Bennison–Soare [1] for dealing with semi-low$_{1.5}$ sets. However, much more is required beyond these methods, and Maass's proof is innovative and technically intricate. Maass's paper will appear in the Transactions of the A.M.S.

Next, Maass and Stob produced an even more complicated extension of the automorphism machinery to prove that if A,B,C,D are r.e. sets, A is a major subset of B (written $A \subseteq_m B$) and $C \subseteq_m D$, then $\mathscr{E}^*(B-A) \cong \mathscr{E}^*(D-C)$. This very pleasing result answered in long open question and extended a decision procedure developed in Stob's dissertation [39] building on an earlier decision procedure by Lachlan. The paper will appear in the Annals of Mathematical Logic.

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